Creative Destruction in a Dynamic Stochastic General Equilibrium Framework: Evaluation Through Calibration and Simulation

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A Thesis
Presented to
The Established Interdisciplinary Committee for Mathematics and Economics
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

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May 2009

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Acknowledgements

The first thing that I think about when reflecting on my thesis experience is this big pink thing on my leg. I'm not sure if I think about it because its painful, or bright, or just really annoying. Two and a half weeks ago my left Achilles decided to call it quits. I thought the thesis process was hard enough, but sure, it could be harder.

Thankfully, everyone around me has been very supportive. First, I would like to thank my parents for coming up to be next to me in the hospital. They made sure I was going to be comfortable in bed for a week before they left. Rick has been a source of invaluable information: teaching me how to shower, how to do wheelies in the wheelchair, and making sure to let me know that falling on my foot is stupid. Rami thank you for always offering to take me to lunch. I may have starved otherwise.

These last four years I have continually relied on Ultimate as an outlet for my academic frustrations. Being a part of this team shaped my undergraduate life. I may have learned just as much with the team as I did from my classes. I could not think of a better collection of people to spend a weekend with. Since their freshmen year when they first came to practice, Bryson and Russell's strange sense of humor and fascination with everything math-econ has continually impressed me. I am sorry that I could not spend my last year playing with Sasha. He always came to practice ready to play as hard as he could, forgetting everything happening outside of practice. Finally, it has been great to have Josh and David on the team with me these last four years. I wish we could have enjoyed our senior year nationals together.

I would also like to thank my advisors for sticking with me even through poor time management. The last week they have been very helpful in making things easy for me and being very understanding. I apologize to the ppw crew for not being able to spend the last few weeks in there with you. Nick, Avery, and KC you guys were good entertainment, while still providing a nice workspace with plenty of funny stuff on the walls.

Finally, I would like to thank all the nice people in admissions. First, they let me into this school. Now they provide a nice place for me to take a break and hang out. Ian has always been supportive of me in all aspects of my life. He encourages me with my thesis, telling me that if he can do it surely anyone can. Thank you for getting me to a lot of the Blazers games safely. Go Blazers. And although Ian and KC always argued about who I was visiting, it was nice to see both of you. KC, swinging with you has been very relaxing. I can not imagine what this semester would have been like without you. Probably much harder. I am continually impressed with you and how our relationship has grown in the last four years.

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Abstract

Increases in the capability of computers and advances in econometrics have enabled dynamic stochastic general equilibriums (DSGE) to gain importance in the field of macroeconomics. The knowledge of creating and simulating DSGE models will become increasingly important for future research in the field. By allowing for a model economy to remain in a dynamic equilibrium state, DSGE models can lead to a more thorough picture of the economy. This can be achieved by a combination of a real business cycle model and endogenous technological advances. Creative destruction fuels the economy through continual innovation. This endogenous process produces both long-term growth and short-term business cycles. While an endogenous process may not replicate the real economy better than a simple exogenous stochastic shock, it provides richer foundations. A more in-depth model allows for more extensive analysis of certain facets of the economy.

Introduction

The two classes of models discussed below each possess certain strengths and weaknesses. Real business cycle theory builds a dynamic stochastic general equilibrium model. Then by using exogenous shocks to productivity, the models are capable of generating artificial business cycles, but often times these are incapable of matching long-term trends. Schumpeterian creative destruction models build from basic microeconomic theory. From these foundations, creative destruction models create an endogenous innovation process resulting in a long-run growth trend. In combining these two classes of models, this thesis will replicate a dynamic general equilibrium model driven by an endogenous technological change mechanism. By constructing such a model, it will be possible to determine whether it is plausible that both economic growth and fluctuations can result from endogenous processes.

Building from basic microeconomic framework, this thesis will replicate a creative destruction model that contains elements of real business cycle models. The driving force of the economy will be twofold. Capital accumulation will provide a stable avenue of growth. The innovation process will produce growth dependent upon the investment in and success of R&D. Finally, the model will be evaluated using data from the U.S. economy. The parameters will be calibrated to match the long-term trend of the economy. Then the model will be simulated. The simulated economy will be compared to the U.S. economy through key economic indicators.

Chapter 1

Literature Review

A quick look at a macroeconomic time series reveals two distinct properties: the long-run growth trend and the short-run fluctuations. Both these properties have important implications for the real world. Until recently, most models have separated the long-run growth from short-run fluctuations.

The traditional economic models focus only on one of the two properties. Keynes (1936) attempted to explain the fluctuations by modeling the sluggishness of price and wage changes. But Keynes's model lacked a basic microeconomic framework. While explaining long-run growth, the Solow (1956) growth model also relied on simple assumptions. Macroeconomists began to question the rationality of economic behavior behind the basic assumptions in the 1970s, leading economists to question the resulting predictions of these models.

Rather than relying on market inefficiencies, real business cycle theory builds on microeconomic foundations of rational and perfectly competitive agents. Building off of traditional growth models, real business cycle models view technological progress as an important piece of the model. Real business cycle theory explains the short-run fluctuations as results of exogenous shocks, most centrally to technology, while remaining in a market-clearing equilibrium state. These shocks were often thought to be productivity shocks. Skeptics of real business cycle theory doubt the major role real business cycle theory places on the relation between exogenous shocks and the economy's response.

Recently, endogenous growth models have combined real business cycle theory with endogenous shocks. The endogenous shocks are often modeled through the use of Joseph Schumpeter's idea of creative destruction.[1] By modeling the shocks endogenously these models attempt to strengthen real business cycle theory.

1.1 Real Business Cycle Theory

Much of microeconomic research builds off an idealized economy. By starting with a perfectly competitive market,[2] it becomes easier to illustrate the effects of an imperfection in the market. Real business cycle (RBC) theory takes a similar approach to the macro economy. Building off microeconomic foundations, RBC theory models the economy in a dynamic equilibrium leading to both long-term growth and short-term fluctuations.

1.1.1 The Basic RBC Model

To build an idealized economy at the macro level, one must start with the microeconomic framework. Following Long and Plosser (1983), the simplest RBC models begins with a neoclassical model of capital accumulation based on the Ramsey (1928), Cass (1965), and Koopmans (1963) model. The economy consists of one individual, Robinson Crusoe. At the beginning of each period Crusoe chooses his consumption, leisure time, and quantity of labor and commodity inputs for final goods production. There are N different commodities produced in the economy. Any commodity can

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[1] Schumpeter (1934).

[2] A perfectly competitive market has several distinguishing characteristics: many agents, homogenous products, low entry and exit barriers, perfect information, and firms maximizing profits.

be used as an input in the production of another commodity. The commodities have a depreciation rate of 100 percent.[3] So, commodities must either be consumed or used in production during each period. Crusoe is constrained by the total amount of commodities as well as the amount of time available. Crusoe chooses his allocation to maximize his utility:

$$U = \text{Max}\, \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t, Z_t) \right],\ 0 < \beta < 1,$$

where $u(C_t, Z_t)$ is the instantaneous utility function of the agent and β is the discount factor, C_t is the $N \times 1$ vector of commodity consumption in period t,[4] and the amount of leisure time in period t is given by Z_t. Crusoe's tastes do not change over time, and are unaffected by the exogenous shocks.

The production function is Cobb-Douglas, with commodity inputs X_t and labor L_t.[5] So, output in any period $t+1$, for good i, is dependent upon the commodities of inputs produced in the last period:

$$Y_{i,t+1} = \lambda_{i,t+1} L_{i,t+1}^{b_i} \prod_{j=1}^{N} X_{ijt}^{a_{ij}},\ i \ = \ 1, 2, \ldots, N.$$

This production function is a standard Cobb-Douglas production function with the addition of the stochastic parameter λ_t. The parameters a_{ij} and b_i sum to one, i.e., $b_i + \sum_{j=1}^{N} a_{ij} \ = \ 1$.

In order to give the model properties mirroring business cycles, we introduce exogenous shocks into the economy. These shocks arise from jumps in technology.[6] Solow (1957) illustrated that the majority of growth in an efficient market with constant

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[3] This implies that any investment must be through the avenue of production. Also, there is no capital in the formal sense.

[4] In a discrete time model, the t subscript always refers to the period.

[5] X_t is an $N \times N$ matrix of commodity inputs. The $(i,\ j)$ element is the quantity of commodity j allocated to production of i.

[6] The assumption that the shocks affect productivity is a feature of most RBC models. The shocks could be introduced in other areas of the economy as well.

returns to scale results from either capital accumulation or productivity increases. The productivity growth has been designated the "Solow residual." Solow showed that these increases in technology account for at least half of output growth in the U.S. since the 1870's. We assume λ_{t+1} represents these technological shocks. It is assumed to be an observable, stochastic, time-homogeneous Markov process.[7]

The completion of the model relies on the two resource constraints: Crusoe's total time available per period, and the amount of commodities available. Let H be the total time available per period. Thus Crusoe's time is constrained by

$$Z_t + \sum_{i=1}^{N} L_{it} = H,\ t = 0, 1, 2, \ldots$$

and the distribution of commodities is restricted by

$$C_{jt} + \sum_{i=1}^{N} X_{ijt} = Y_{jt},\ j = 1, 2, \ldots, N;\ t = 0, 1, 2, \ldots.$$

In each period Crusoe must decide the allocation of time and commodities with the information given in time t, (C_t, L_t, Z_t, X_t). His decision will be based on his marginal rates of substitution of labor for consumption. Given the utility function: $u(C_t, Z_t) = \theta_0 \ln Z_t + \sum_{i=1}^{N} \theta_i \ln C_{it}$, the optimal distribution can be solved analytically.

The optimal quantities are as follows:

$$C_{it}^{*} = \left(\frac{\theta_i}{\gamma_i}\right) Y_{it},$$

$$Z_{it}^{*} = \theta_0 \left(\theta_0 + \beta \sum_{i=1}^{N} \gamma_i b_i\right)^{-1} H,$$

$$X_{ijt}^{*} = \left(\frac{\beta \gamma_i a_{ij}}{\gamma_i}\right) Y_{ij},$$

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[7] A time-homogenous Markov process is "memoryless", i.e., given $\lambda_t, \lambda_{t-1}, \lambda_{t-2}, \ldots$, then the conditional distribution of λ_{t+1} depends only on λ_t.

$$L_{it}^{*} = \beta\gamma_i b_i \left(\theta_0 + \beta\sum_{j=1}^{N}\gamma_j b_j\right)^{-1} H,$$

where $\gamma_j = \theta_j + \beta\sum_{i=1}^{N}\gamma_i a_{ij}$.

The optimal consumption and input quantities result in two principles of behavior:

> (1) The portion of the total available stock of a commodity allocated to a given employment (consumption) is an increasing function of its productivity in that employment (consumptive value). The same principle applies to the allocation of the time (H) available in a period.
>
> (2) The amount of a commodity (or time) allocated to each of its productive employments and to a positively valued consumption are all increasing functions of the total available amount of the commodity (or time).[8]

The second of these two principles is essential in studying business cycles. If a technological shock causes the output of commodity i to be unexpectedly high, then production that requires commodity i as an input will also be unexpectedly high in that period. If we assume that the commodity i is used as an input in several sectors, the propagation of cycles become apparent. The productivity in those sectors will increase, resulting in higher output in several sectors. Not only is this effect seen across sectors, but forward in time as well. By maximizing his lifetime utility, Crusoe will attempt to smooth his consumption.[9] So, if there is a large positive productivity shock, Crusoe will use more commodities as inputs to production transferring the positive shock to future periods. Interestingly, Crusoe can not change these results because the shock cannot be foreseen.

While this model seems overly simplistic, it helps us understand business cycles. Even in this simple model, it is possible to create short-term fluctuations resulting

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[8] Long and Plosser (1983).

[9] Crusoe cannot smooth consumption through the usual avenue of capital accumulation. Instead, he allocates more output back into production. He can do this every period increasing total production in the long-term.

from productivity shocks. All the while, the economy remains in a state with the markets clearing. The first generation of RBC models excels only in the creation of cycles. Looking more closely at the cycles, we struggle to compare the model's cycles with real economic cycles. The reliance on exogenous factors creates random cycles not necessarily mirroring the real economy. Also, in attempting to explain deep recessions, RBC theory fails. A deep recession would require a large decrease in productivity, because that is the only avenue for a cycle. A large negative productivity shock like this seems implausible. So, while the general creation of dynamic equilibrium cycles is a useful tool, it lacks characteristics replicating the real economy. And while a refined RBC approach may be capable of modeling the economy there are still more general pitfalls.

1.1.2 Criticisms of the Basic Model

As with any new area of research, many objections have been raised to RBC theory. Summers (1986) and Mankiw (1989) raise four objections to the theory. The first objection examines the technology shocks. The shocks given in the model seem so large that they are unrealistic, even though calibrated using the Solow residual. Empirical studies have shown that the Solow residual can be misleading. Variations in the Solow residual are not necessarily technology shocks, as it has been supposed. Instead, some of the exogenous factors imbedded in the Solow residual may result from changes caused by money, interest rates, or government spending. These changes may increase efficiency throughout the economy without affecting productivity. Without such large technological shocks, the model may not be able to account for the short-run fluctuations.

The second objection examines the propagation mechanisms. The model relies on laborers' intertemporal substitution of labor and leisure. If the laborers do not substitute labor between periods, the propagation of cycles is minimized. But, labor must fluctuate considerably to match the cycles of the real economy. If labor does not fluctuate then wages must, and labor needs to be sensitive to these fluctuations. The model also predicts cyclicality of wages. None of the relationships needed in the model are reflected in the economy. This criticism has been supported empirically. The studies find that the elasticity of labor substitution is small, meaning laborers do not substitute intertemporally in the manner assumed in the model. The wages also do not have the cyclical properties necessary for the model to behave properly.

Thirdly, the model omits any monetary aspect. There is considerable evidence that monetary shocks have real effects as well.[10] By lacking any monetary features, this channel is absent completely. Finally, many of the dynamics do not mirror the real US economy. While in the model output dynamics appear similar, other variables such as consumption and hours worked are modeled poorly. These objections are all raised against the basic RBC model, but there have been many extensions since it was first proposed.

1.1.3 Extensions

There have been many extensions of the basic model. Stadler (1994) and Rebelo (2005) give an excellent summary of these extensions and RBC models in general. Many of these attempt to address these concerns. The first objection will be addressed in the remaining pages. By attempting to build the technological progress into the model, we can avoid relying on the empirical Solow residual. This process will be created by a Schumpeterian creative destruction process, discussed below. Otherwise, the extensions focus on bolstering the model in four general categories: labor market, money, government, and trade.

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[10] Romer and Romer (1990) provide sufficient evidence.

Labor Market

In the RBC model above employment and wages are heavily procyclical. In reality, these macroeconomic variables are less volatile than the model predicts. Hansen (1985) provides a model in which labor is indivisible (i.e., a worker can be hired for a given amount of time or not hired at all). In his model, total hours worked become far more variable relative to productivity when compared to the U.S. economy. Cho and Cooley (1988) extend this model to allow labor to choose both the hours worked and whether to be employed. This provides better performance when examining the productivity and hours worked relationship.

The correlation between productivity and output is necessarily high in the RBC model because cycles are caused by productivity shocks. This correlation is more difficult to see in reality.[11] Several things have been tried attempting to properly model this relationship. Bencivenga (1992) investigates the role of preference shocks[12] within the households. She concludes that such shocks may be more plausible if viewed as productivity shocks to home production.[13] Benhabib et al. (1991) create a model which includes home production and shocks to that sector. Shocks to the home sector cause fluctuations in the labor supply in the production sector. Critics of home production models have two basic critisisms. The real economy shows positive correlation between investment in market and nonmarket capital over cycles, whereas, the model predict a negative correlation.[14] The main cause of business cycles is voluntary movement between market and home activities, which is also unrealistic. This has been seen as a result of cycles, not a cause.

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[11] There is great difficulty in measuring productivity changes in the real economy. Many different things can increase productivity outside of technology

[12] Preference shocks are shocks to preference parameters in the model, such as the discount factor β.

[13] Home production is output produced by the household sector such as preparation of meals, cleaning, and transportation services. This is thought to be quite large, between twenty and fifty percent of GNP.

[14] These criticisms arise from Greenwood and Hercowitz (1991).

In order to account for unemployment, a labor market search must be introduced into the basic RBC model. Mortensen (1990) and Andolfatto (1996) introduce the search process in an RBC model. This addition results in two interesting changes. First, changes in employment respond only to permanent increases in productivity, rather than temporary increases. Second, productivity shocks have an increased effect on output. By introducing a non-clearing labor market, Danthine and Donaldson (1991) create involuntary unemployment.

Money

Money has also been introduced into the RBC framework. King and Plosser (1984) create a financial sector within the RBC model. Financial services can be produced more rapidly than goods with the inputs of capital and labor. Financial services are used in production of final good and purchases of final goods. Both firms and households save time by using these services. An increase in output caused by a shock to productivity will increase demand for financial services. This model predicts correctly that these services will fluctuate with output.

Other monetary extensions include nominal rigidity. Yun (1996) incorporates monopolistic competition and nominal price rigidity. The results demonstrate stronger co-movement between inflation and output better than flexible-price models, mirroring reality. Introducing nominal rigidity helps bridge the gap between Keynesian models and RBC models. Many view the lack of such features as a major weakness of RBC models.

Government

Several papers have introduced government in order to see the effect of new policies. Many of these models provide weak policy implications because of the homogeneity of the population. So, in order to better evaluate government policies that framework must be changed. Other models have included governments in which spending also follows an exogenous stochastic process. These models result in small changes to output volatility. Baxter and King (1993) include distortionary taxes that finance government purchases. This model concludes that aggregate output will decrease further resulting from an increased spending financed by taxes.

Open Economy

There have been many papers that create an open economy RBC model. These models attempt to explain trade patterns within the RBC framework. Several models have been able to duplicate several stylized facts. First, the counter-cyclical movement of the balance of trade and second, the positive correlation of the relative prices of exports and imports with the trade balance (Backus et al. (1994)). Also, Finn (1989) models the correlation between savings and investment in open economies.

1.2 Schumpeter's Creative Destruction

The basic idea of creative destruction was first proposed by Schumpeter (1934). Aghion and Howitt (1992), Segerstrom et al. (1990), and Grossman and Helpman (1991)[15] extend Schumpeter's ideas to incorporate modern economic modeling techniques. This model endogenizes the rate of technological progress. Through that endogenization the growth becomes a random process which could be cyclical.[16]

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[15] The model below follows the Aghion and Howitt paper. The Segerstrom et al. (1990) model differs slightly from the one here. They assume a deterministic amount of time between innovations. Also, they include a more expansive intersectoral structure. Grossman and Helpman (1991) also create a similar model building on both of the previously mentioned papers. Their model contains many products each of which has its own quality ladder. Thus, innovations are sector specific.

[16] The goal of this seminal creative destruction model is not to explicitly model short-run fluctuations. The model focuses on long-run growth, which happens to be cyclical by its random nature.

1.2.1 The Basic Creative Destruction Model

In the basic model, there are three sectors: producers of final goods, research and development (R&D) firms, and consumers. The R&D firms supply a single type of intermediate good to the producers of the final goods. The basic model lacks any capital accumulation. The economy consists of L individuals each endowed with one unit of labor. Unlike the RBC model above, the laborers do not have the labor-leisure tradeoff. The rate of time preference, r,is assumed to equal the interest rate. Since there is no capital accumulation considered, all of the output is consumed. In this model t represents time, and given an output of $y(t)$, the laborers have preferences

$$u(y) = \int_0^\infty y(t)e^{-rt}dt.$$

The laborers either produce intermediate goods or do research giving the equation

$$L = x + n$$

with n being the labor used in research, and x labor used in the production of intermediate goods. The production of the final good depends on the input of the intermediate good, x, giving total output:

$$y_\tau = A_\tau x^\alpha, \ 0 < \alpha < 1.$$

Here τ represents the number of previous innovations that have occurred and indexes the productivity parameter A_τ.

In this model, the efficiency of production of final goods is dependent upon the technological parameter A. Innovation in the R&D sector raises the technological parameter. Each new intermediate good that replaces the old good increases A_τ by the constant factor, $\gamma > 1$. Innovations arrive through a random process. This process is a Poisson process with an arrival rate of $\lambda \cdot n$. The productivity of the research, λ, is determined exogenously.

An R&D firm that succeeds in innovating past the current producer of the intermediate good becomes a monopoly in producing that intermediate good. Any good that is not the newest innovation becomes obsolete in the process. This assumes that the newest innovation can be perfectly substituted for the obsolete good with no cost.[17] This assumption makes the model more tractable, but it stretches the real world observations.[18]

In the research sector, the expected value of an hour of research must be equal to the value of an hour in the production of final goods (given by the wage, w_τ). This gives the arbitrage condition that determines the amount of labor devoted to research:

$$w_\tau = \lambda V_{\tau+1}$$

where w_τ is the wage and $V_{\tau+1}$ the discounted expected payoff to becoming the next monopoly. The labor market equation combined with the arbitrage equation forms the foundations of the Schumpeterian model.

Diving further into the arbitrage equation, we can begin to determine the expected payoff of innovation through the asset equation. The value of an innovation equals the profit flow available, $\pi_{\tau+1}$, minus the value of capital lost once the monopoly is replaced, $\lambda n_{\tau+1} V_{\tau+1}$. The expected income given by an innovation in a unit time interval, $rV_{\tau+1}$. Equating these two values allows us to determine the value of $V_{\tau+1}$;

$$rV_{\tau+1} = \pi_{\tau+1} - \lambda n_{\tau+1} V_{\tau+1}$$

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[17] This assumption allows us to only consider one intermediate good, x. We know the newest intermediate good is always used in final production, therefore avoiding any aggregation problems that could result from different vintages of intermediate goods.

[18] Schumpeter recognized that how private firms and social firms react to new technology will differ. Private firms tend to wait until the old capital wears out before replacing it, unless the profit motivation is large enough to encourage faster adaptation. Social firms will tend to adapt sooner recognizing the increases in efficiency available and the benefit to the economy.

And solving for $V_{\tau+1}$ yields

$$V_{\tau+1} = \pi_{\tau+1}/(r + \lambda n_{\tau+1}).$$

Examining the denominator more closely, we see that it is the interest rate adjusted by the arrival rate of new innovations. So, the greater the arrival rate, the smaller the payoff is of each innovation. This equation lies at the heart of the creative destruction process.

After defining π_τ the model will be complete. Since we assume perfect competition in the final goods sector, we know the price of x will be equal to the marginal product of the input x in producing the final good. We derive the demand curve from the production function above giving $p_\tau(x) = A_\tau \alpha x^{\alpha-1}$. The profit of any single firm is $\pi_\tau = p_\tau(x)x - w_\tau x$. So, the equilibrium values can then be found by maximizing profit with respect to x.

Without detailing the steady-state equilibrium, we will jump to the implications for long run growth.

$$y_\tau = A_\tau F(L - \hat{n})$$

is the steady state value of output with $\hat{n}$ being the equilibrium value of n. This implies

$$y_{\tau+1} = \gamma y_\tau.$$

Remembering that τ here is the number of innovations, we see the time between two innovations is random. This means the time path of $\ln y(t)$ will also be a random step function, each step being equal to $\ln \gamma$. Finally, the time between steps will be exponentially distributed with parameter $\lambda \hat{n}$. So, the expected growth rate will be equal to $\lambda \hat{n} \ln \gamma$.

Social versus Private Growth

A second method of analysis compares the growth rate with the socially optimal growth rate. A social planner will take into account both the private and public benefit of any innovation. In order to do this, the social planner will maximize the expected present value of consumption, $y(t)$. By maximizing the utility function above, the social planner concludes that $g^{\star} = \lambda n^{\star} \ln \gamma$. There exist several differences between the socially optimal level $n^{\star}$ and the equilibrium level $\hat{n}$ which maximizes the firms profits. These result from several differences in what they view as optimal.

First, the social planner considers the benefit of a innovation to be of infinite duration[19] while the private firm only sees its benefit lasting until the next innovation. Secondly, the monopolist does not allocate the output properly. Finally, the business stealing effect encourages research firms to become the monopolist. While, the social planner acknowledges that the return of the previous innovation is lost. The first two effects lead to insufficient research under laissez-faire growth while the last will generate too much growth. Thus we cannot conclude whether growth will be too slow or too fast. This misalignment of social and private optimal growth is an interesting result of Schumpeterian creative destruction models not illustrated in other models.

1.2.2 Criticisms of the Basic Model

Aghion and Howitt (1998) consider several shortcomings of the model. They call attention to three limitations. First, many of the assumptions made are used to ensure a steady-state equilibrium leading to a balanced growth path. One of the most binding assumptions is that of perfect substitutability in the intermediate good market. The notion that final good producers will immediately stop using the old innovation and fully transfer all production to using the new innovation is not reflected

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[19] Benefits are of infinite duration because the innovation being considered is a prerequisite for further innovations.

in reality. By restricting the model, many important questions about the economy cannot be addressed. A second imperfection highlights the unconventional research process. When an innovation succeeds it essentially builds on the previous knowledge by a constant factor. This is reflected in the increase in A. But, the concept that ideas all have the same effect on production seems unrealistic. It would better reflect the real innovation process if a learning curve was also introduced. Finally, the model lacks any institutions. These mainly affect the R&D process. The true process of research becomes much more complicated in reality. There is not one innovator who then becomes a monopoly until replaced. Many financial and institutional restraints force R&D to follow certain tendencies not reflected here. These limitations all result from the simple foundations of the model.

1.2.3 Extensions

Several extensions of the basic model have provided solutions to these problems. Aghion and Howitt (1998) provide an excellent summary of these extensions. Howitt and Aghion (1998) add multiple sectors in the intermediate good sector as well as capital accumulation to their original model. By adding multiple sectors it requires the innovations to be specific to each sector. This results in the steady state being described by constant growth, not the stochastic growth above. Multiple sectors with individual uncertainties about innovation create a continuous flow of innovations when aggregated. Thus, by the law of large numbers, there is no longer that uncertainty in the macro economy.[20] The model also includes capital accumulation where intermediate goods are produced by capital instead of labor. This allows final goods to be consumed, invested in R&D, or used in the production of capital goods. These additions make the model more realistic and model growth through both innovation and capital accumulation, an obvious trend in the economy.

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[20] But looking more closely at the micro economy, there is still a stochastic nature. The aggregation causes the randomness to disappear.

Howitt (1999) follows Young (1998) in attempting to minimize the "scale effect" apparent in creative destruction models. The scale effect results from an increase in population. This increase leads to a rise in the rate of technological process.[21] In the long-run, the output per person grows faster. Empirical studies have not supported this theory.[22] So, both vertical and horizontal innovations are included in the model. Vertical innovations are the same innovations as in the model above and horizontal innovations increase the number of goods in the intermediate good market. By increasing the possibilities for innovation, the model dissipates the increases in productivity making output per person constant in the long run, mitigating the scale effect.

Walde (1999) studies household behavior in greater detail in a similar model. He decentralizes the economy and relaxes the assumptions behind the utility function. Households in his model are risk-averse and invest part of their earnings in R&D. This minimizes a couple of the market failures mentioned above. The households internalize some of the market failures because the social and private goals become more aligned. So while households receive some benefits from innovation, the monopolies still gain from discovering the newest technology. One market failure remains: the monopoly rents motivating research firms still only account for part of the social gain from an innovation.

Along the same line of this research, Judd (1985) and Romer (1990) explore horizontal innovations. Innovations result in new classes of goods rather than more productive goods of the same nature. They find that larger markets induce faster technological progress and therefore more growth. Jovanovic and Rob (1990) study the endogenous determination of major discoveries and their refinement. Their study

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[21] Kremer (1993) creates a model which supports this idea. He supposes that every individual has a chance of being smart enough to invent independent of population. Thus, higher population yields higher growth in technology.

[22] Jones (1995) supports the claim that the scale effect is false.

differs in that they explore the effect on specific industries rather than the aggregate effects.

The strength of the creative destruction model lies in its ability to endogenize technological progress. If we remember the RBC model, one of the primary weaknesses was the reliance on exogenous technological progress. By combining the strengths of each of the models, it may be possible to build a better endogenous growth model that models both long-run growth and short-run fluctuations effectively. This model will be examined in the next section.

1.3 Endogenous Growth Cycles

The creative destruction model above has been extended to examine cycles in the economy. Bental and Peled (1996) develop a model that produces separate innovation and capital accumulation cycles. These cycles result from incentives to do research or accumulate capital. They modify the search process making it more difficult for more advanced economies to innovate by decreasing the probability of innovations when the currently level of technology is high. This changes the tendencies of the cycles over time, mirroring industrialized economies better. The changes in the innovation process make the model very appealing.

1.3.1 The Basic Cyclic Model

The framework of the model corresponds to the Diamond overlapping generations growth model. N agents are born in each period t. Agents live two periods, during the first of which they provide labor inelastically, consume, and save. Savings are invested in a fixed number of firms. In the second period the agents rely on their previous savings for consumption. Thus, saving is determined by a utility maximization problem. The utility function is

$$u(c_t, \tilde{c}_{t+1}) = \ln(c_t) + \frac{\beta}{1-\beta}\mathbb{E}_t\left[\ln(\tilde{c}_{t+1})\right],\ 0 < \beta < 1.$$

$(c_t, \tilde{c}_{t+1})$ is the consumption in the first and second period respectively by an agent born at t. $\tilde{c}_{t+1}$ is unknown at period t because it depends on the firms profits. Because of the log utility function, a young agent will earn a wage of w_t then save βw_t regardless of the distribution of the rates of return on savings. Additionally, each agent will equally diversify his investment between each firm if their returns are identical and independently distributed.

Firms partake in two activities in each period using the savings from the previous period. At the beginning of each period, firms engage in a costly search for technology. Using the remaining capital, firms hire the labor and produce output by utilizing technology.[23] Before hiring labor, firms know their levels of capital and technology. The firms produce output given by the usual Cobb-Douglas production function with k units of capital, l units of labor, and a technology indexed by $\theta \geq 1$:

$$y = A\theta k^{\gamma} l^{(1-\gamma)},\ 0\ <\ \gamma\ <\ 1$$

The firms can either use the technology they discovered, or the minimum economy wide technology. The minimum technology is the best of all technologies discovered in the previous period. After the period ends, the firm distributes the profits earned to the agents who invested in the firm.

Remembering that labor is provided inelastically we take the wage rate equal to w, which is determined by the post-search demand.[24] This yields the profit-maximizing

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[23] The capital can be thought of as both physical and financial capital. The firms use the capital to conduct the technology search, pay labor, and as factors of production.

[24] The equilibrium wage rate is determined post-search and the labor-market clearing equation will be given after that process has been defined.

level of employment:

$$l^*(k, \theta, w) = k \cdot \left(A\theta \cdot \frac{1-\gamma}{w} \right)^{1/\gamma}$$

Substituting this level of labor into the equations for output and firm's profit gives:

$$y(k, \theta, w) = (k \cdot \theta^{1/\gamma}) \cdot A^{1/\gamma} \left(\frac{1-\gamma}{w} \right)^{(1-\gamma)/\gamma}$$

$$\pi(k, \theta, w) = \gamma(k \cdot \theta^{1/\gamma}) \cdot A^{1/\gamma} \left(\frac{1-\gamma}{w} \right)^{(1-\gamma)/\gamma}$$

Given those equations, we can now establish the search process that maximizes the profit for each firm.

Each of the I firms begins each period with capital q and a default technology, θ_0. Before beginning production, the firm can choose to search for a better technology. This search involves the firm drawing randomly from a technology pool. The firm is not guaranteed to find a technology better than the current available technology. This pool is a fixed and known Pareto distribution, $H : [1, \infty] \rightarrow [0, 1]$. Bental and Peled choose the Pareto distribution for two reasons. First, it is unbounded above allowing for infinite technological improvements. Second, the density of the distribution is declining while doing so slowly. This allows for the idea that better technologies are less frequently discovered. By modeling the technologies as such, it allows for incentives to research to remain in more advanced economies. Also negating the possibility of a no-growth equilibrium. Thus, the probability of finding a technology less than or equal to θ is

$$H(\theta) = 1 - \theta^{-\lambda}, \theta \geq 1, \lambda > 0.$$

By drawing from the pool, the firm incurs a cost of α. The firm can either choose the last technology drawn, continue searching, or use the default technology. For simplification, each firm takes the wage rate as given, ignoring the effect further searches will have on the labor market. Thus, after they finish the search process the firm combines the remaining capital with labor to produce. To maximize profits, firms attempt to maximize the level of technology and capital available after the search, i.e., maximize $E\left[\tilde{k}\cdot\tilde{\theta}^{1/\gamma}\right]$, where $\tilde{k}$ and $\tilde{\theta}$ are the amount of capital and technology available after the search process.[25] Let $\theta^*(k)$ denote the threshold technology level which satisfies these properties.

After the search process is finished, it is possible to find the aggregate output values in equilibrium. First, we must define the market-clearing equilibrium wage rate:

$$w = (1-\gamma)A(1/N)^{\gamma}\left(\sum_i \theta_i^{1/\gamma}k_i\right)^{\gamma}$$

where (k_i, θ_i) are the post-search values of capital and technology. Substituting the wage rate into individual firm's output and aggregating yields:

$$Y = \sum y_i(k_i, \theta_i, w) = AN^{(1-\gamma)}\left(\sum_i \theta_i^{1/\gamma}k_i\right)^{\gamma}$$

This ends our summary of the model. For ease of presentation, the dynamics of the model will not be described mathematically. Bental and Peled solve for two locus curves. From there they are able to determine the path of the economy given the value of parameters. We will describe these conclusions more generally.

The cycles in the model stem from the assumption about the default technology available. If the search process yields a significant technological discovery, firms will be less likely to search in the periods that follow. The default technology will be greater than the threshold required to stop searching. Without searches occurring, firms will not have to finance the search process. This results in capital accumulation.

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[25] Looking at the profit function this maximization becomes more apparent. The firm will attempt to maximize expected profit in the search: $E\left[\pi(\tilde{k},\tilde{\theta},w)\right] = \gamma A^{1/\gamma}\cdot E\left[\left(\frac{1-\gamma}{w}\right)^{(1-\gamma)/\gamma}\right]\cdot E\left[k\cdot\theta^{1/\gamma}\right]$. Since the only part changed by the search process is $E\left[k\cdot\theta^{1/\gamma}\right]$, the firm will maximize that value during the search process.

This is one of two types of cycles. As capital accumulates, the rate of return of further capital accumulation decreases. After a certain amount of capital has been accumulated the firms find it worthwhile to reallocate some resources to R&D. The firms do this because the expected value of R&D given a certain amount of capital becomes greater than the marginal value of that capital. And then begins another cycle of search for technology. After each cycle, the economy is becoming more and more advanced. As the economy advances, large technological improvements become more difficult. This can lead to *quasi-steady-states*. In this state, firms engage in search for technology, but the default technology is not improved upon by any firm. These quasi-steady-states increase in both duration and likelihood as an economy advances.

The long-run growth is characterized by two paths dependent upon the parameters. If $\gamma + 1/\lambda > 1$ then the economy generates positive expected growth rates. The growth in this situation results from successful search phases and no-search phases with a finite duration. With $\gamma+1/\lambda < 1$ the economy converges to a no-growth steady-state given any initial conditions. This results from extended no-search phases with savings not capable of generating positive growth.

1.3.2 Extensions

By modeling the technological process as improvements, rather than levels, the return to investment in R&D decreases as technology improves.[26] This presents an alternate picture of R&D, especially when examining industrialized economies.

Walde (2002) presents a model based on the Aghion and Howitt (1992) model which also examines growth cycles. He argues that both the growth rate of an economy and the fluctuations are results of an endogenous process. The process for which cycles occur is similar to the creative destruction model presented. As technology in-

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[26] The process in the Aghion and Howitt (1992) model uses technological levels (by assuming that a fixed investment generates a fixed distribution of improvements without regarding current levels).

creases, the return on investment in R&D decreases thus increasing the rate of capital accumulation. His paper illustrates the power of a simple creative destruction model when applied to a textbook Ramsey continuous time growth model.

Matsuyama (1999), Matsuyama (2001) also explores long-run growth through cycles. His model is based off the lab equipment model of Rivera-Batiz and Romer (1991). His papers argue for two sources of growth, one from factor accumulation and another based on innovation, similar to the model above. The innovations in his models are more deterministic, causing long-run growth with deterministic cycles. Freeman et al. (1999) require accumulation of research experience and slow adaptation of new technologies. This model also results in a similar growth and cycle pattern. Finally, Francois and Lloyd-Ellis (2003) illustrate a similar result through the bunching of innovations.

Many of the models above find R&D to be countercyclical, usually as a direct result of the assumptions. Walde and Woitek (2004) determine that R&D is generally procyclical in G7 countries. This follows Fatas (2000) who finds a similar result for the United States. The stochastic nature of R&D in previous models creates an all or nothing expenditure in R&D. Redding (2002) focuses on technological lock-in. This distinguishes fundamental and secondary technological developments with a path dependence mechanism. His model predicts continuous R&D investment, which may be procyclical. Walde (2005) closely analyzes the portfolio decision problem to create procyclical R&D. Individuals make investment decisions with their savings to finance either capital accumulation or R&D. The driving force behind R&D becomes procyclical dividend payments, making R&D expenditure procyclical.

Chapter 2

Evaluation of DSGE Models

In order to evaluate the strength of a dynamic stochastic general equilibrium (DSGE) model, we must be able to determine its implications for the economy. Traditional growth models could be solved analytically. The resulting growth patterns and steady-state equilibrium could then be compared to those of the real economy. But, unlike those models, non-linear dynamic discrete-time stochastic models cannot always be solved analytically. Thus, a new method of evaluation was developed.

First, the model must be simplified to satisfy several restrictions of the simulation process. Then, the model is either calibrated, giving values to parameters from previous econometric studies, or the parameters are estimated using various statistical techniques. With the parameter values determined, the model can then be simulated. During simulation the model economy is subjected to one or more exogenous stochastic shocks. Since the steady state is already exogenously specified in the model, only the deviations from the steady state are evaluated. After the simulation, the model economy is compared to a de-trended real economy using standard deviations, correlations, and serial correlations of various aggregate macroeconomic indicators.

2.1 Specifying the Model

Several important properties provide the foundation for any DSGE model. First, there must exist a non-stochastic steady state. The steady state values of the endogenous variables can be defined as a function of the various parameters. Often these steady state values are specified to match the observed economy. The design of the model, therefore, defines the long-term growth trend of the model economy, in the absence of the stochastic shocks.

Secondly, a DSGE model consists of a collection of first-order and equilibrium conditions derived from the specification of the model and microeconomic foundations. These include the current and future expectations of endogenous variables as well as current values and future expectations of exogenous stochastic shocks. The solution can then be described through a set of equations relating current endogenous variables to the past state of the economy and current shocks.

2.2 Determining Parameter Values

2.2.1 Calibration

The term calibration refers to the setting of the parameter values to match an economy based on preexisting evidence. The calibration method was first used by Kydland and Prescott (1982) to analyze their real business cycle model and then became the standard when real business cycle models were being simulated. In order to calibrate a model, the parameters of the model are given either values based on empirical microeconomic evidence or values to match the data, e.g., the discount rate (β), used in the utility function for many DSGE models. Using the calibrated parameters, the economy is then simulated.

Calibration's main strength is the ease of using pre-determined parameter values, assuming there have been prior studies on those parameters. Calibration also does not require as much computational power, minimizing the time required for the simulation. The method of calibration was the standard way of determining parameter values in the first generation of DSGE simulations. But, as econometric methods advanced and processing power increased, new methods were developed.

These new methods highlight the weaknesses of calibration. The parameters in the model must be limited to values that have been studied, or could be determined from econometric analysis. This makes the use of more abstract parameters difficult. Unknown parameters can be guessed then tested for robustness. A model can only be as complex as the known parameter values allow.

2.2.2 Econometric Estimation

The method of econometric estimation refers to several distinct techniques used to calculate parameter values based on the underlying data to which the simulation is applied. The generalized method of moments, minimum distance estimation, maximum likelihood, and Bayesian methods are the most commonly used. This section will focus on the Bayesian estimation method because it is usually seen as the strongest estimation method and the simulation program Dynare employs this method.[1]

Bayesian estimation consists of specifying prior distributions, means, and standard deviations of the parameter values. A log-likelihood function is derived using the priors and the specified DSGE model usually through a Kalman filter. The Kalman filter allows the non-linear parameter equations to be evaluated by the likelihood function. Finally, the Metropolis-Hastings algorithm is used to estimate the posterior distribution of the parameter value.[2]

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[1] Canova (2007) provides a detailed review of the different approaches not discussed here.

[2] Griffoli (2007).

Several key advantages make Bayesian estimation the strongest of the various methods. The Bayesian estimation uses the complete DSGE model when estimating the likelihood functions. This ensures an accurate representation of the prior distributions with respect to the model. By defining the prior distribution, that allows for more detailed specification of unknown parameters. The use of priors also ensures the likelihood function will have appropriate curvature while minimizing spikes by matching the prior.

Two main difficulties weaken the attractiveness of Bayesian estimation. If arbitrary priors are introduced without motivation[3] the posterior may be overly biased towards that prior. Thus, priors must ideally be based on some real information about the parameter. Additionally, estimating posterior distributions for parameters is computationally intensive. Besides the obvious implication of increased simulation time, it also makes replication of simulations more difficult. This may decrease the transparency of simulations using Bayesian estimation.

2.3 Simulation of the Model using Dynare

Several different programs are available for simulating DSGE models. "Dynare is a powerful and highly customizable engine, with an intuitive front-end interface, to solve, simulate and estimate DSGE models."[4] Dynare's interface is easy to learn, making it ideal for use in this thesis. Outside of simulating models, Dynare is capable of:

- Computing the steady state of a model.
- Computing the solution of deterministic models.

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[3] This motivation should come from information known about the parameter in any form, whether it be data, theory, or previous studies.

[4] Griffoli (2007).

- Computing the first and second order approximation to solutions of stochastic models.
- Estimating parameters of DSGE models using either a maximum likelihood or a Bayesian approach.
- Computing optimal policies in linear-quadratic models.[5]

In order for a model to be simulated using Dynare, it must be written in the proper form. Analysis of a DSGE model examines deviations around the steady state caused by shocks. Thus, the model must eventually return to the steady state after the shock wears off. This requires that the model must be stationary. A stationary model has no long-term growth trend, meaning all variables will return to their steady state values in the long-run.[6]

The model must also be linearized. A linear model is written in terms of deviations from the steady state, allowing for easier analysis. Dynare gives you the option of doing this by hand, or allowing the program to linearize the model for you. When done by hand, the model is first written in terms of the log-deviations from the steady state. Then, first-order Taylor approximations of the equations are taken to finish the linearization.

Dynare uses the same method as would be done by hand, but adding the second-order Taylor approximation is usually not done by hand. Adding the second-order approximation allows both the first and second moments of shocks to remain in the model. The first moments disappear when expectations of the shocks are taken. But, the variance of future shocks remains after taking expectations, thus deepening the model. This is a necessary addition for certain models.

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[5] Griffoli (2007).

[6] This is achieved by first determining the long-term growth trend of an economy, if any. Then, dividing out the growth rate from any variable that experiences it in the long-run.

Steady state values must also be calculated using the parameter values. However, these steady state values do not need to be exact, because Dynare will calculate the exact steady state values regardless. But, they need to be approximately close to the exact values in order for Dynare to complete the calculations. Finally, parameter values must be entered. These can be done through one of the two methods discussed above.

Once the model is linearized, Dynare begins to simulate the model.[7] The specified exogenous shocks are fed into the model. The model is simulated over a certain number or pre-specified periods. After the simulation occurs, Dynare outputs several types of information to evaluate the model, discussed below. The output consists of:

1. A summary of the types of variables in the model
2. Matrix of covariance of exogenous shocks
3. Policy and transition functions
4. Moments of simulated variables
5. Correlations of simulated variables
6. Autocorrelations of simulated variables
7. Impulse Response functions for each of the endogenous variables with respect to each exogenous shock

2.4 Evaluating the Simulated Economy

Once the economy is simulated there are two main avenues of analysis. Impulse response functions provide a visual and quantitative analysis of the effect of shocks on the endogenous variables. These can be compared to policy changes or other

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[7] The mathematical details of the Dynare process can be found in Appendix C.

shocks observed in the real economy, and the resulting effects. The impulse response functions help to visualize the effect of policy decisions. They also illustrate the effects of shocks over time. This is useful when exploring lags of shocks or the adjustment time of specific variables. However, impulse response functions cannot be used to determine the relationship between endogenous variables, instead various statistical measures are used.

In order to relate the simulated economy to the real one, several key statistics can be compared. After de-trending various aggregate macroeconomic indicators,[8] the standard deviations, correlations, and autocorrelations can be compared. The most common macroeconomic indicators evaluated are output, consumption, investment, wages, and employment. This evaluation allows the researcher to see what parts of the economy the model recreates accurately and what parts it is unable to recreate. It also shows the relationship between variables, in the form of correlations. There is no standard measure for what makes a good model or a bad model. Often models are trying to measure certain relations in the economy, so those correlations will be looked at more closely when evaluating. The difficulty in determining the quality of a model presents difficulties when attempting to evaluate policy decisions based on that model.

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[8] They must be de-trended because the simulated economy is stationary whereas the real world is not. This is usually done with a Hodrick-Prescott filter.

Chapter 3

The Model Economy

The model presented in this chapter has evolved from the creative destruction ideas originally presented by Schumpeter (1934). In a continuous-time framework, Howitt (1999) and Aghion and Howitt (1998), build an endogenous growth model with capital accumulation using Schumpeter's ideas. This chapter follows Nuno Barrau (2008) who develops a discrete-time version of those models.

Similar to the Aghion and Howitt model described in Section 1.2.1, final good production firms and intermediate good production firms populate the economy. Final goods are produced in a perfectly competitive market with intermediate goods and labor used as production inputs. Final goods can be consumed, used in the production of intermediate goods, or invested in the financial sector. Each intermediate good sector utilizes a specific technology developed in the research sector. The incumbent monopolist[1] produces the intermediate good using capital rented from households. The incumbent intermediate good producer gains monopolistic profits by selling its product.

The creative destruction process fuels the growth of the economy through productivity advances. Entrepreneurs research innovations specific to a particular in-

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[1] The incumbent monopolist developed the most advanced technology currently available for that sector in a previous period.

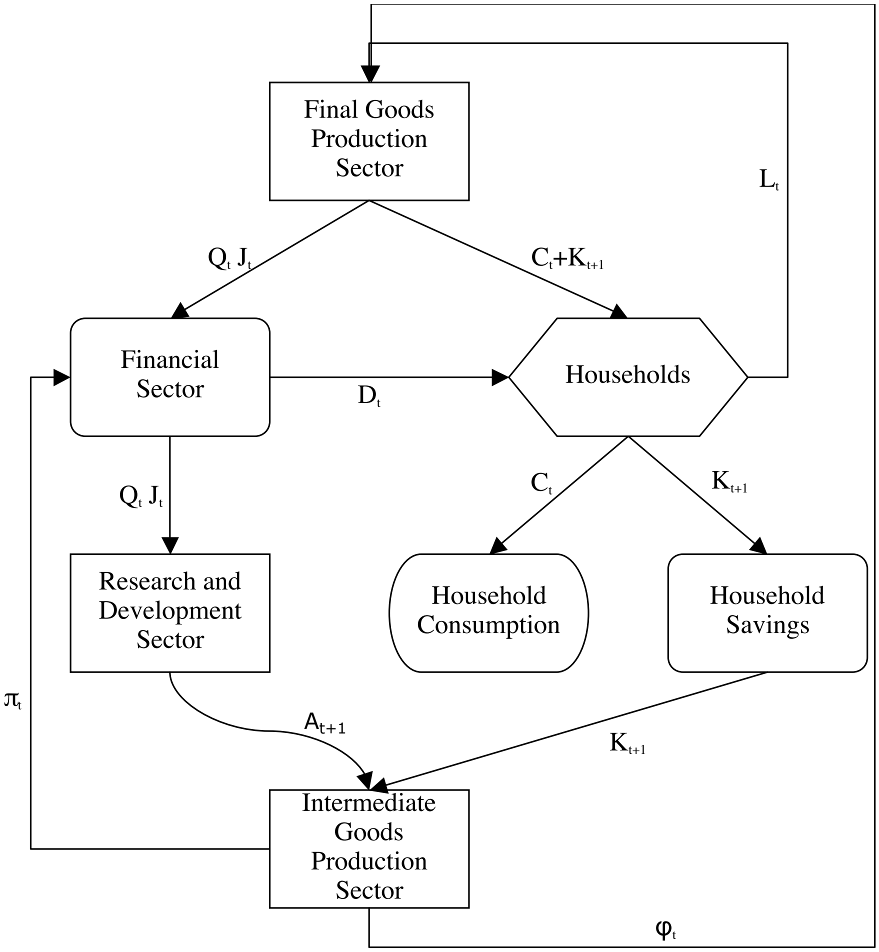

Figure 3.1: A flowchart of the economy

termediate good sector using investments from the financial sector. If the research is successful then the entrepreneur replaces the incumbent monopolist producer in that intermediate good sector. The new producer utilizes the advanced technology to gain the monopoly profits. This new innovation also has spill-over effects increasing innovation by entrepreneurs in other sectors in later periods.

Entrepreneurs receive funding from the financial sector. By assumption, the sector is risk neutral and profit maximizing. The financier loses all the funds of entrepreneurs unsuccessful at creating a new innovation. But, they receive the profits of the successful entrepreneur's intermediate good production firm which is then paid out as dividends to households.

Households are infinitely lived and risk-averse. They maximize consumption over their lifetime. Households provide labor inelastically for final goods production. Consumption is financed by wages, renting capital, and investment dividends. The households will be described in detail first. This will be followed by the final and intermediate production firms. The innovation and financial sectors are then introduced. Finally, the model's equilibrium and steady-state will be illustrated.

3.1 Households

Households consist of a population of infinitely lived laborers, L_t. Each laborer forms one household. Letting C_t be aggregate consumption and $c_t = C_t/L_t$ be consumption per labor unit. The household's expected utility over its lifetime is given by

$$\mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \frac{(c_{t+i})^{1-\gamma}}{1-\gamma}, 0 < \ \beta \ < 1, \tag{3.1.1}$$

where β is the discount factor. The households maximize expected utility subject to the budget constraint:

$$C_t = w_t L_t + D_t + (1+r_t)K_t - K_{t+1} + \xi_t. \tag{3.1.2}$$

The laborers receive wages $w_t L_t$ from the production of final goods, financial firms pay dividends D_t, and capital is rented to firms at r_t. Since the economy is simulated using U.S. macroeconomic data, ξ_t is used as a proxy for the current account balance. The exogenous term ξ_t guarantees the budget constraint with respect to actual data.[2]

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[2] The economy considered here is a closed economy. Thus, ξ_t cannot be modeled as current account balance but rather must be an exogenous shock.

For simplicity, $\hat{\xi}_t = \frac{\xi_t}{(A_t L_t)}$will follow an independent autoregressive process: [3]

$$log(\hat{\xi}_{t+1}) = (1-\rho_\xi)log(\xi) + \rho_\xi log(\hat{\xi}_t) + \sigma_\xi \epsilon_{\xi,t+1}, \ |\rho_\xi| < 1 \ , \ \sigma_\xi > 0, \tag{3.1.3}$$

where ϵ_t^ξ is assumed to be a normally distributed i.i.d. process with mean zero and variance unity. The value ξ does not change over time. The labor force is assumed to have a growth rate following the autoregressive process:

$$log(G_{t+1}^L) = (1-\rho^L)log(G^L) + \rho^L log(G_t^L) + \sigma_L \epsilon_{L,t+1}, \ |\rho^L| < 1 \ , \ \sigma_L > 0, \tag{3.1.4}$$

where ϵ_t^L is assumed to be a normally distributed i.i.d. process with mean zero and variance unity.

3.2 Final Goods Production Firms

Final goods can be used for consumption, as capital input to intermediate good production, or for investment in research. The production of final goods requires labor and an aggregate of the intermediate goods, of which there are J_t different kinds. Total output, Y_t, is a continuum of intermediate goods of measure J_t given by the production function

$$Y_t = \int_0^{J_t} A_t(j)\varphi_t^\alpha(j)(L_t/J_t)^{1-\alpha}dj, \tag{3.2.1}$$

where $A_t(j)$ is the productivity level of the j^{th} intermediate production firm and $\varphi_t(j)$ is the productivity-adjusted flow of intermediate good output.[4] The value $N = L_t/J_t$ is assumed to be constant so that the number of intermediate products is proportional

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[3] A variable x_t that has been divided by the effective labor unit, $A_t L_t$, will be denoted by $\hat{x}_t$.

[4] The only use for intermediate goods is as input into final goods production. Thus the output $\varphi_t(j)$ is used wholly as inputs into final good production. The output is in productivity-adjusted units which is then multiplied by $A_t(j)$ to get physical units of input. The resulting exponent on $A_t(j)$, if the intermediate good production function is substituted in, is $(1-\alpha)$.

to the labor-force size. By holding N constant, this ensures that total output does not increase with a growth in product variety. This assumption mitigates the scale effect defined in Section 1.2.3.

The production of final goods occurs in a perfectly competitive market. Thus, firms set marginal products equal to the factor prices. Differentiating with respect to the intermediate good production yields the price for intermediate good j:

$$p_t(j) = \alpha A_t(j)(\varphi_t(j)/N)^{\alpha-1} \tag{3.2.2}$$

and differentiating with respect to labor unit yields wages of

$$w_t = (1-\alpha)y_t, \tag{3.2.3}$$

where $y_t = \frac{Y_t}{L_t}$ is output per labor unit. Thus, the income share of output devoted to workers is constant.

3.3 Intermediate Goods Production Firms

Intermediate goods are used only in the production of final goods. Each intermediate good is produced by a distinct firm both indexed by j. The production of intermediate goods requires capital and also depends on the productivity of each producer, $A_t(j)$. This gives a production function

$$\varphi_t(j) = (u_t K_t(j))/A_t(j), \tag{3.3.1}$$

where $K_t(j)$ is the capital input for the j^{th} sector. And u_t is the utilization rate of that capital.[5] The exogenous variable u_t accounts for shocks to capital utilization such as

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[5] The capital utilization rate could also be modeled through microeconomic foundations for a more complete model.

shifts in aggregate demand, a rise in commodity prices, or changes in monetary or fiscal policy. As the economy advances, intermediate good production becomes more capital intensive, guaranteed with the division by $A_t(j)$.

The model is again simplified by assuming u_t to follow an autoregressive stochastic process:

$$log(u_{t+1}) = (1-\rho^u)log(u) + \rho^u log(u_t) + \sigma_u \epsilon_{u,t+1},\ |\rho^u| < 1\ ,\ \sigma_u > 0. \tag{3.3.2}$$

Once again, ϵ_t^u is assumed to be a normally distributed i.i.d. process with mean zero and variance unity.

Nuno Barrau (2008) determines the costs for an intermediate production firm equal $(r_t + \delta)K_t(j)$, where r_t is the rate of interest and δ is the depreciation rate. Substituting in $K_t(j)$ from the production equation gives a marginal cost of $A_t(j)(r_t + \delta)\varphi_t(j)/u_t$. Intermediate good producers then receive a price $p_t(j)$ determined by Equation 3.2.2. Thus, marginal revenue equals $p_t(j)\varphi_t(j) = A_t(j)\alpha N^{1-\alpha}\varphi^{\alpha}(j)$.

Since marginal revenues and marginal costs for each intermediate producer differ only by the value of $A_t(j)$, they all choose to produce at a level

$$\varphi_t = \varphi_t(j) = u_t\hat{k}_t N \tag{3.3.3}$$

where $\hat{k}_t = K_t/(A_t L_t)$ and A_t is the average level of technology across all sectors J_t. Revisiting the production equation for final output per labor:

$$y_t = \int_0^{J_t} A_t(j)\varphi_t^{\alpha} L_t^{-\alpha} J_t^{\alpha-1} dj = A_t\,(u_t\hat{k}_t)^{\alpha}. \tag{3.3.4}$$

The incumbent monopolist producers will then maximize profit such that marginal revenue equals marginal costs. That equality and the substitution of $\varphi_t = u_t\hat{k}_t N$

yields an equilibrium interest rate

$$r_t = \alpha^2 u_t^{\alpha} \hat{k}_t^{\alpha-1} - \delta. \tag{3.3.5}$$

The monopolists then earn a profit per unit labor of

$$\pi_t(j) = A_t(j)\alpha(1-\alpha)(u_t\hat{k}_t)^{\alpha} N. \tag{3.3.6}$$

This yields a monopolists' mark-up of $\alpha(1-\alpha)$ percent of the revenues.

3.4 Research and Development Sector

Entrepreneurs provide the driving force for the economy. At the beginning of each period one entrepreneur per intermediate good begins research devoted to that good. The entrepreneur relies on investment from the financial sector and the current technological knowledge. If the entrepreneur succeeds in developing a new innovation, that raises $A_t(j)$ to the technological frontier of the last period, A_{t-1}^{max}.[6] A successful entrepreneur will then replace the incumbent monopolist in the production of that intermediate good.

Innovations arrive with a Poisson arrival rate of $n_t(j)$.[7] The arrival rate of innovations depends on the amount of capital invested, $Q_t(j)$, the current technological frontier, and an exogenous parameter $\bar{\lambda}_t$:

$$n_t(j) = \sqrt{\frac{2Q_t(j)}{\bar{\lambda}_t A_t^{max}}},\ n_t(j)\ \geq 0. \tag{3.4.1}$$

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[6] The technological frontier represents the most advanced technology in all intermediate good sectors, $A_t^{max} \equiv \max\{A_t(j) \mid j \in [0, N_t]\}$.

[7] With the discrete-time framework it is necessary to assume that the probability of two successful innovations in one period is negligible.

The innovation process experiences decreasing returns to scale. It is assumed that A_t^{max} grows at a constant rate.[8] As the economy become more technologically advanced, through increases in A_t^{max}, future innovations require higher costs.

The resource cost of research are represented by $\bar{\lambda}_t$. Once again this exogenous variable follows an autoregressive stochastic process:

$$log(\bar{\lambda}_{t+1}) = (1-\rho^{\lambda})log(\lambda) + \rho^{\lambda}log(\bar{\lambda}_t) + \sigma_{\lambda}\epsilon_{\lambda,t+1},\ |\rho^{\lambda}| < 1\ ,\ \sigma_{\lambda} > 0. \tag{3.4.2}$$

The innovation shock, ϵ_t^{λ}, is a normally distributed i.i.d. process with mean zero and variance unity. An Innovation shock represents wide-ranging changes to the economy increasing efficiency in production such as general purpose technologies, or improved managerial systems. The shocks will be negative in order to represent a decrease in the cost of research.

Let the value of becoming a monopolist at time t in sector j be defined as $v_t(j)$, a function of $A_t(j)$ and $n_t(j)$. The value is equal to the profit obtainable in time t, given an innovation raising $A_t(j)$ to A_{t-1}^{max}, plus the discounted expected value of profit if no innovation occurs in $t+1$. This yields:

$$\begin{aligned} v_t(A_t(j), n_t(j)) &= v_t(A_{t-1}^{max}, n_t(j)) \\ &= A_{t-1}^{max}\alpha(1-\alpha)(u_t\hat{k}_t)^{\alpha}N + (1-n_t(j))\frac{\mathbb{E}_t[v_{t+1}(A_{t-1}^{max}, n_{t+1}(j))]}{(1+r_t)}. \end{aligned}$$

The value of innovation will fluctuate randomly as the arrival rate, $n_t(j)$, changes. Given that the benefits and costs of innovation in each sector differ only by $n_t(j)$, the probability of innovation will be equal across sectors in equilibrium, i.e., $n_t(j) = n_t$.

Technology then evolves according to the equation: $A_{t+1} = n_t A_t^{max} + (1-n_t)A_t$. The technology frontier, A_t^{max}, grows as a result of the spillovers produced from

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[8] This growth rate is assumed in order to match the data. The technology does not necessarily reach that level immediately though.

innovations. This growth is assumed to be at a constant rate, g. The growth of the frontier represents the evolution through continual advance of technology in specific sectors of the economy.

3.5 Financial Sector

The availability of credit in a capitalist system is essential. The financial sector plays two roles in the economy. First, the financial sector is able to efficiently allocate resources to different sectors of the economy. Second, the financial sector allocates the risk inherent in funding entrepreneurial endeavors. In each period, the financial sector chooses an amount $Q_t(j)$ to invest in each sector. Since the probability for success is the same for each sector,[9] the financial sector invests a total of $Q_t J_t$ resources into research. The financial sector receives the profits of a successful innovation, π_t. And if the entrepreneur does not succeed, the financial sector loses its investment.

The financial sector will attempt to maximize its flow of dividends over the infinite lifetime. The dividends are given by total profits minus total investment, $D_t \equiv (\pi_t L_t - Q_t J_t)$. The financial sector will then maximize

$$\mathbb{E}_t \sum_{i=0}^{\infty} \left(\prod_{l=t+1}^{t+i} (1+r_l)^{-1} \right) D_t, \tag{3.5.1}$$

subject to the profits of intermediate firms, the probability of a successful innovation, and the technological frontier.

The financial sector's maximization problem yields the first order condition:

$$n_t = \mathbb{E}_t \left[\frac{\alpha(1-\alpha)\,(1+g)\,(u_{t+1}\hat{k}_{t+1})^{\alpha}\, G_t^L\,(1 - A_t/A_t^{max})\,N}{(1+r_{t+1})\,\bar{\lambda}_t} \right]. \tag{3.5.2}$$

---

[9] $Q_t(j) = Q_t$.

This equation illustrates the financial sector trade-off when making their investment decision. Investments increase if the expected profits increase, through an increase in intermediate good output. Investments decrease if the expected discount rate rises or the economy moves closer to the technological frontier.

3.6 Equilibrium and Balanced Growth Path

In order to simulate the model, it must first be made stationary. Letting $a_t(j) \equiv A_t(j)/A_t^{max}$ be the distance to the technological frontier,[10] rescaling $\lambda_t = \frac{\bar{\lambda}_t}{N}$ such that $\hat{q}_t \equiv (Q_t J_t)/(A_t L_t) = \frac{1}{2a_t}\lambda_t n_t^2$ is the investment per labor unit, and setting $G_t^L \equiv L_{t+1}/L_t$ achieves stationary. Thus the Euler equation for the households can be written as:

$$1 = \beta \mathbb{E}_t \left[\frac{\hat{c}_t^{\gamma}\,(1+r_{t+1})\,a_t^{\gamma}}{\hat{c}_{t+1}^{\gamma}\,G_t^L\,a_{t+1}^{\gamma}\,(1+g)^{\gamma}} \right]. \tag{3.6.1}$$

When in equilibrium, the market is characterized by a balanced growth path with output, consumption, investment, and capital per labor-unit growing at the constant rate g. This can be shown through the evolution of the two state variables, $\hat{k}_t$ and a_t

$$\hat{k}_{t+1} = \frac{a_t\left((u_t\hat{k}_t)^{\alpha} + (1-\delta)\hat{k}_t - \hat{c}_t - \hat{q}_t - \hat{\xi}_t\right)}{a_{t+1}(1+g)G_t^L} \tag{3.6.2}$$

and

$$a_{t+1} = \frac{n_t(1-a_t)+a_t}{1+g}. \tag{3.6.3}$$

The two state variables have constant steady state values $\hat{k} = \left(\frac{\alpha^2 u^{\alpha}}{r+\delta}\right)^{1/(1-\alpha)}$ and $a = \frac{n}{n+g}$.[11] The economy progresses according to these two processes: the amount of capital accumulation occurring and the average distance to the technological frontier,

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[10] This gives a comparative measure of the technology level for every sector. Each sector is either at the maximum technological level, A_t^{max}, or behind by a certain amount depending on when the last successful innovation occurred.

[11] In the steady state, $r = \frac{G^L(1+g)^{\gamma}}{\beta} - 1$ and $n = \frac{g}{2}\left(-1 + \sqrt{1 + 4\frac{\alpha(1-\alpha)(u\hat{k})^{\alpha}G^L}{g(1+r)\lambda}}\right)$.

which measures the maximum technology level and the average level of technology used in production.

The constant steady state values implies that K_t, A_t, and L_t grow at the rate $(1+g)G^L$, $(1+g)$, and G^L respectively. This yields a growth rate of $(1+g)G^L$ for Y_t, C_t, and Q_tJ_t when the economy is in its steady state.

Chapter 4

Data, Simulation, and Results

In this chapter the model will be put to data attempting to replicate Nuno Barrau (2008). This entails finding the proper parameter values, simulating the economy, then evaluating using the proper data. The first section will describe the data. The relations between the model variables and the real macroeconomic aggregate data will be explained. This includes transforming the data and determining the relationship between the simulated variables and the observed ones. The next section details the simulation process. First, the parameter values are explained. Then the model is linearized, the steady state is found, and it is input into Dynare. The final section examines the numerical results as well as the impulse response functions. The simulated model is compared both to the data and the simulation results from Nuno Barrau (2008). From there, conclusions can be made about the merit of the model.

4.1 Real Data and Simulated Variables

4.1.1 U.S. Economy Data

The data used for evaluation of the model comes from the OECD national accounts and the World Development Indicators (WDI). The macroeconomic aggregates come from OECD while the population data comes from WDI. All the data are yearly data from the United States in the period 1960 to 2005. The gross domestic product (GDP) at constant prices proxies for output, Y_t. The sum of private and government consumption is total consumption, C_t. Government and private gross fixed capital formation and changes in stocks represents total investment, I_t^T, described below. Finally, the total labor, L_t is the total population aged 15-64.[1]

4.1.2 Transformation of Data

In order for the real economy to be compared to the simulated one, the relationship between them must first be defined. Let dy_t, dc_t, di_t^T, and dL_t be the observed series where $dx_t = \log(x_t/x_{t-1})$ and $x_t = \frac{X_t}{L_t}$. Thus, dx_t is the growth of X_t per labor unit and dL_t is the growth of the labor force.

The real economy has a trend growth rate along with the variable growth rates. This trend must be accounted for in the model. Rather than using a Hodrik-Prescott filter or another de-trending device, the growth rate is calculated and the trend growth rate is subtracted. For example, the labor force is represented as:

$$dL_t = \log(L_t/L_{t-1}) = \log(G^L) + \tilde{G}_{t-1}^L. \qquad (4.1.1)$$

The growth trend parameter, G_L, is set such that the simulated economy matches the real economy. The log-linearized deviation[2] of the growth of labor from the steady state is represented by $\tilde{G}_{t-1}^L$, discussed in Section 4.2 and Appendix A. Similarly,

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[1] Nuno Barrau (2008) uses total population aged 15-64 because it is not dependent upon the economic conditions whereas the labor force is endogenous.

[2] Mathematically this equals: $\tilde{x}_t = \log(x_t/x)$ where x is the steady state value.

output and consumption are detrended:

$$\begin{aligned} dy_t = \log(y_t/y_{t-1}) &= \log(1+g) + \log(\frac{\hat{y}_t a_t}{\hat{y}_{t-1} a_{t-1}}) \\ &= \log(1+g) + \alpha(\tilde{a}_t - \tilde{a}_{t-1}) + \alpha(\tilde{y}_t - \tilde{y}_{t-1}) \end{aligned} \tag{4.1.2}$$

$$\begin{aligned} dc_t = \log(c_t/c_{t-1}) &= \log(1+g) + \log(\frac{\hat{c}_t a_t}{\hat{c}_{t-1} a_{t-1}}) \\ &= \log(1+g) + (\tilde{a}_t - \tilde{a}_{t-1}) + (\tilde{c}_t - \tilde{c}_{t-1}). \end{aligned} \tag{4.1.3}$$

Finally, investment must be considered. In the model there are two types of investment: investment into research and development and capital accumulation. Thus, total investment can be defined as: $I_t^T = K_{t+1} - (1-\delta)K_t + Q_t J_t$. Investment will equal the value of capital formation, $\hat{i}_t$, plus the value of research, $\hat{q}_t$. So, $di_t^T = \log(i_t^T/i_{t-1}^T) = \log((\hat{i}_t + \hat{q}_t)/(\hat{i}_{t-1} + \hat{q}_{t-1}))$ is the growth rate of total investment with

$$\hat{i}_t = \frac{K_{t+1} - (1-\delta)K_t}{L_t A_t} = \frac{G_t^L (1+g) a_{t+1}}{a_t} \hat{k}_{t+1} - (1-\delta)\hat{k}_t. \tag{4.1.4}$$

This yields an observed investment of

$$di_t = \log(1+g) + \frac{\hat{q}}{\hat{q} + \hat{i}}(\tilde{q}_t - \tilde{q}_{t-1}) + \frac{\hat{i}}{\hat{q} + \hat{i}}(\tilde{i}_t - \tilde{i}_{t-1}), \tag{4.1.5}$$

where $\hat{q}$ and $\hat{i}$ are the steady state values of those variables.

The relationship between the model and the data is complete. First, each variable must be transformed such that it comes in the form of dx_t. The U.S. data is also transformed. This transformation requires adjusting for the size of labor and finding the growth rates. A simulated variable dx_t can then be compared directly with the corresponding variable coming from the U.S. data. With the description of the data complete and the relationship to the model fleshed out, the simulation process follows.

4.2 Simulation of the Model

From this point on this thesis departs slightly from the methods used by Nuno Barrau (2008). Rather than using Bayesian estimation to determine the parameter values, the calibration approach is taken using parameter values as estimated by Nuno Barrau (2008). Thus, this section is an attempt to replicate the simulations done in that paper without estimating the parameters. Of course, because there are several stochastic exogenous shocks, a replication cannot be exact.

4.2.1 Parameter Values

Nuno Barrau (2008) follows a two-step process to determine the parameter values. First, known parameters are calibrated. These parameter values fall into two categories: the growth trend parameters discussed above and previously studied parameters like the discount factor. These parameters are reported in Table 4.1. Then, the rest of the parameters are estimated using the Bayesian techniques discussed in section 2.2.2. The estimation results will be taken as the true values of the parameters for the simulation model in this thesis. Table 4.2 reports the parameter values that were estimated in Nuno Barrau (2008).

Table 4.1: Calibrated Parameters

Variable	Description	Value
α	Output Elasticity of Intermediate Good	0.35
β	Discount Factor	0.99
g	Long-term Growth of Output	0.0191
G^L	Long-term Growth of Labor Force	1.0135
ξ	Current Account Deficit	$-0.016 * \hat{y}$

The discount factor, β, is calibrated to a value typically found in microeconomic literature. The other four parameters in the table above are set to guarantee the simulated economy matches the real one. In equations 4.1.1, 4.1.2, 4.1.3, and 4.1.5,

Table 4.2: Estimated Parameters

Variable	Description	Estimated Mean
n	Steady-state Rate of Creative Destruciton	0.017
γ	Risk-aversion Coefficient	2.479
δ	Depreciation Coefficient	0.115
λ	Innovation Coefficient	1.0389
u	Capital Utilization Coefficient	0.8
	Autoregressive Parameters	
ρ_λ		0.946
ρ_u		0.783
ρ_ξ		0.949
ρ_L		0.928
	Shock Variances	
σ_λ		0.247
σ_u		0.053
σ_ξ		0.289
σ_L		0.001

the necessity for these parameters is illustrated. Finally, the parameter ξ is used to assure the budget identity of the simulated economy matches that of the real economy. The difference results from the current account deficit, which averages approximately 1.6% of GDP in the period. These parameters are calibrated the same in both this simulation and the model.

Table 4.2 shows the partial results of the estimations done in Nuno Barrau (2008). The estimation procedure used follows a two-step estimation process. First, the parameters were estimated using the maximum-likelihood process. This estimation gave approximate means and standard errors. Then using those values, the parameters were estimated using the Bayesian techniques.

The prior distributions for all of the variances of the shocks were defined to be inverted Gamma distributions. This allows for a positive variance with a large domain. Similarly, the risk aversion coefficient and capital depreciation coefficients were taken to be inverted gamma distributions. Unlike the other parameters, the capital depreciation mean is not estimated using maximum likelihood. It is taken to be 10%, a value used widely in economic literature. A beta distribution is used for the autoregressive parameters which ranges between 0 and 1. Finally, the probability of an innovation is also assumed to be a beta distribution because of the domain required. These distributions were used in Nuno Barrau (2008) to estimate the parameters. The estimation procedure returns a mean, 5%, and 95% values.

Without estimating the parameters, there is no procedure allowing for distributions of parameter values to be used in the simulations. So, the mean is taken to be the exact parameter value. This may lead to some of the differences in the simulation replication below. Some of the standard deviations in the estimations are much larger for certain parameters. By being unable to replicate these it may affect certain parts of the model more than others. Unfortunately, these changes are difficult to track because of the stochastic nature of the equations most of the parameters are involved in. After calibrating all the parameter values, the model must be linearized and the steady state found.

4.2.2 Preparing Model for Simulation

In order to approximate the solution to the model, the model must be linearized. This can either be done by hand or through Dynare. In his paper Nuno Barrau (2008) simulates the model that was linearized by hand. He does this in order to make the transformation from simulated variable to observed variables easier. By log-linearizing the model around the steady state, the model becomes a Taylor approximation where all the equations are linear functions of the log-deviations of the variables. A more complete description and examples of log-linearization can be found in Appendix A. Simplifying the model in this way, the steady state and the deviations from that state can be described separately.

The steady state values of the endogenous variables are also required for simulating the model. Dynare will calculate the exact steady state values as part of the simulation. But, Dynare requires approximate steady state values for all the endogenous variables as initial values.[3] The steady state values of the non-linearized model will need to be either calculated exactly or approximated whether the model is linearized or not.

The actual process of inputting the model into Dynare is relatively simple. The input file (mod-file) used is shown in Appendix B. The variables are declared first. Then the parameters are declared along with their values. The steady state values of the non linearized variables are calculated by Matlab. The model is then declared. Predetermined variables are given the notation $x(-1)$. And expectations are automatically taken of variables with the notation $x(+1)$. Then, the initial values for the endogenous variables are given. And finally, the shocks are defined along with the number of periods the simulations should run for.

4.3 Numerical Results

In this section the simulation results will be presented. The simulations that were replicated will be presented first. These will be compared to the simulations in Nuno Barrau (2008). Then, the simulations presented in that paper will be discussed. These simulations will be compared to the data and a generic RBC model, as done in the Nuno Barrau (2008). Because the model is attempting to be replicated, the simulations done for the purpose of this thesis will not be compared directly to the data. Any discrepancies leading to a more accurate model, in the sense of replicating the data, were accidental.

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[3] There may be some confusion here. The steady state values for a linearized model will be zero by definition. But, in the process of linearization, the old steady state values may remain in the model. These are the values that will need to be calculated. If the model is not linearized, the steady state values will need to either be calculated or approximated. Dynare will calculate the exact steady state value in either case.

4.3.1 Replicated Simulation Results

This section will focus on the goal of replicating the simulations presented in Nuno Barrau (2008). The broader implications and merit of the model, more of interest to the true economist, is discussed in the next section. Table 4.3 compares some of the simulation results. Unfortunately, these values are all of those available in Nuno Barrau (2008). A more in-depth analysis could be done with full simulation results. The replicated simulation closely mirrors the actual simulation.[4] It is difficult to look at the simulation results and see what were errors and what are caused by the random fluctuations. But, some general observations will be made.

Table 4.3: Simulation Results

	Replication	Nuno Barrau (2008)
Std. Deviations		
dy_t	0.0201	0.0197
dc_t	0.0161	0.0158
di_t	0.0779	0.0734
dL_t	0.0027	0.0028
Correlations		
(dy_t, dc_t)	0.9198	0.9248
(dy_t, di_t)	0.8388	0.8720
(dy_t, dL_t)	-0.0290	-0.0230
(di_t, dc_t)	0.6438	0.6996
(dy_t, dy_{t-1})	0.1325	0.1342
(dy_t, dy_{t-2})	0.0898	0.1224

All but two of the replication values fall within five percent of the actual values. The values that are off by more than five percent are the correlation between output and investment and the second lag autocorrelation of output. These differences cannot be accounted for by random error. The error causing the disparities in these values could not be determined. Because gazing at these broad statistical measures does not

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[4] The random nature of the simulation required that the ten simulations be averaged, the average values are presented.

give us total insight into the accuracy of the model, we will also examine the impulse response functions.

The impulse response functions show the response of four key variables from the two key shocks, the capital utilization shock and the innovation shock. Three of the variables are the observed variables used to compare to the U.S. data. The fourth variable is a_t or the distance from the technological frontier. Comparing Figures 4.1 and 4.2, the impulse response functions of the capital utilization shock, they look very similar (accounting for any scale differences).

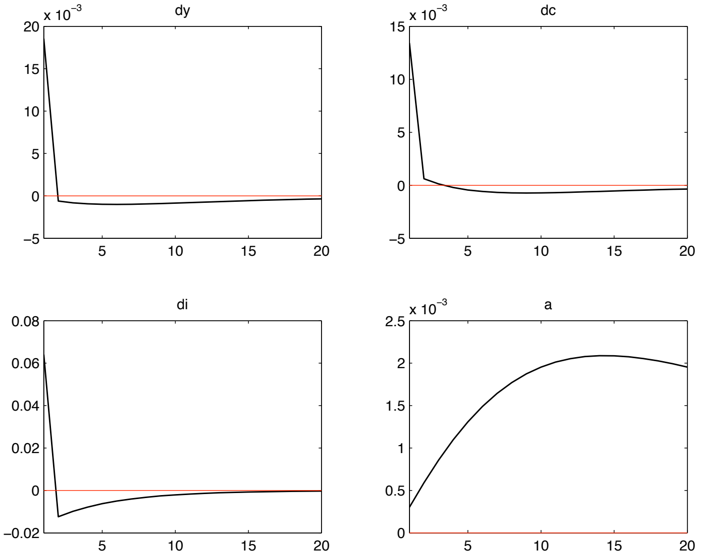

Figure 4.1: Impulse Response Function for a Capital Utilization Shock, u

Figures 4.3 and 4.4 model a negative shock on lambda. A negative shock on lambda will increase the arrival rate of innovations. An increase in the arrival rate of innovations increases the distance from the technological frontier, a_t. This distance represents the difference between the average technology and the maximum possible technology. So, with a slower rate of innovation, this distance will become greater. This also increases the incentive for innovation since a new innovation will gain higher profits. These impulse response functions show the success of the simulations.

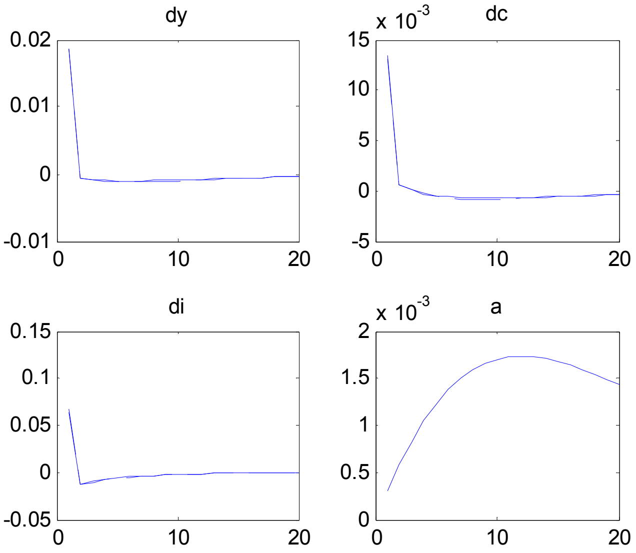

Figure 4.2: Impulse Response Function for a Capital Utilization Shock, u, from Nuno Barrau (2008)

The goal of replicating and simulating this model was twofold: learn the process of solving and simulating a DSGE model, and successfully replicate a Schumpeterian-RBC model so extensions could be modeled. In the process of learning how to solve and simulate a DSGE model many tools must be utilized. Every model has unique features but these general tools are needed to determine the merits of any such model. In the process of simulating the model presented above, these methods have been employed with success.

In order to successfully replicate a DSGE such as the one above, many of the facets of the model must be learned. Without building a full model from scratch, some of the details may be difficult to determine. This makes recreating a simulation even more difficult. Some of the discrepancies may be results of unsolved simulation errors. Other factors, outside of errors, that may lead to the differences discussed results from the stochastic nature of the model and different or unknown parameter values[5].

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[5] Discussed in Section 4.2.1 above.

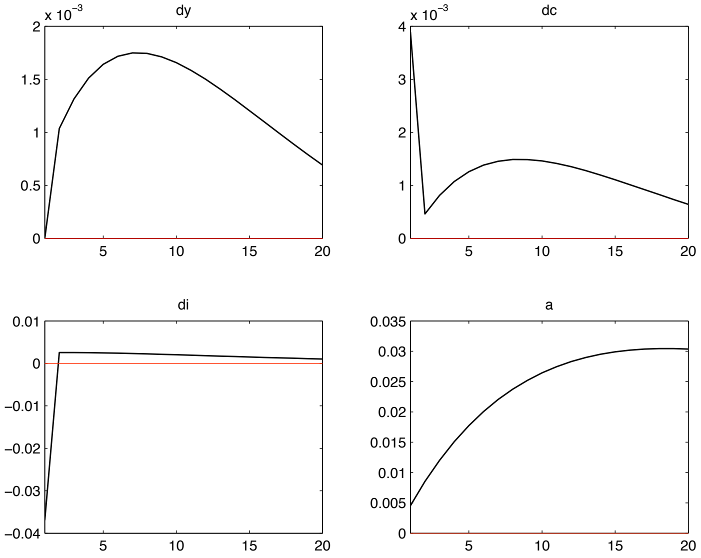

Figure 4.3: Impulse Response Function for an Innovation Shock, λ

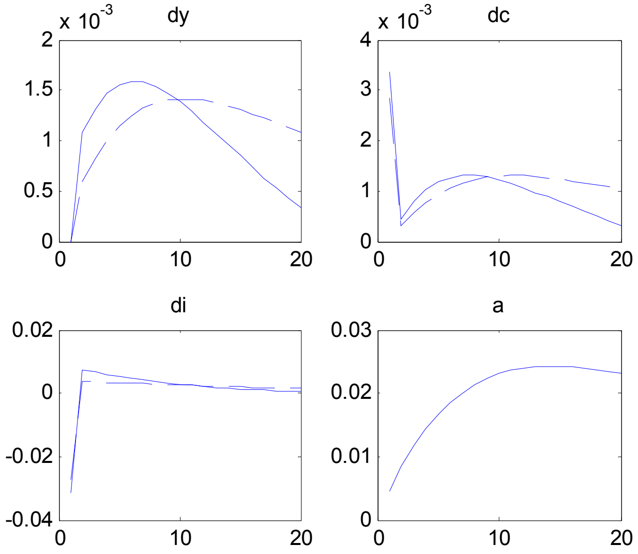

Figure 4.4: Impulse Response Function for an Innovation Shock, λ, from Nuno Barrau (2008)

4.3.2 Conclusions of the Model

Because the model was successfully simulated, either model can be compared to the U.S. economy. So, for ease, we will examine the conclusions made by Nuno Barrau (2008) using those simulation results. In his paper he also simulated a simple RBC model with an exogenous technology shock.[6] The RBC model is very similar to the model presented without the endogenous productivity growth process. This model was used to evaluate the merits of endogenous technological progress versus an exogenous process. These conclusions will be discussed first.

Table 4.4: Simulation Results from Nuno Barrau (2008)

	Data	Nuno Barrau (2008)	RBC
Std. Deviations			
dy_t	0.0185	0.0197	0.0198
dc_t	0.0146	0.0158	0.0159
di_t	0.0707	0.0734	0.0757
dL_t	0.0034	0.0028	0.0027
Correlations			
(dy_t, dc_t)	0.8823	0.9248	0.9252
(dy_t, di_t)	0.8661	0.8720	0.8692
(dy_t, dL_t)	-0.1835	-0.0230	-0.0190
(di_t, dc_t)	0.5954	0.6996	0.6977
(dy_t, dy_{t-1})	0.3143	0.1342	0.1585
(dy_t, dy_{t-1})	-0.0642	0.1224	0.1449

Figure 4.4[7] illustrates the difference in the persistence of the observed variables after an innovation shock. The RBC model shows more persistence in the observed variables. But, the shock also takes longer be reflected in an increase in output in the RBC model. Looking at the statistical interpretation in Table 4.4 of each simulated model, we can see they very closely reflect each other. The models appear to be very similar in the simulations.

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[6] In the RBC model the shock is simply a technology shock to the productivity parameter commonly found in production function. There is no endogenous process creating an increase in productivity.

[7] The dashed line is the RBC model while the solid line is the Schumpeterian model

The simulated model is able to partially replicate a couple of the key statistics presented. The output-investment correlation as well as the standard deviations of some of the observed variables are relatively consistent with the data. This reflects the strength of the microeconomic framework and endogenous growth. While long-term growth is not discussed heavily, the model is also able of replicating this. The ability of the model to do this is simply by definition of the parameters. But, few models are capable of replicating both long-term growth and short-term fluctuations.

The question may then be asked, why go to the trouble of specifying the model such that the shocks are results of an endogenous process? The Schumpeterian model provides both a richer foundation and a richer sense of productivity shocks. By beginning with a microeconomic framework, the assumptions of the model provide a more realistic picture of the economy. This allows for a closer examination of certain pieces of the economy if desired. Furthermore, in the traditional RBC model the shocks are seen as increases in technology used for production. The Schumpeterian models allows for other types of productivity increases as well. These may come in the form of more efficient business practices, financial innovations allowing greater access to credit for entrepreneurship, or a lowering of entry costs.

Both of the simulated models were not capable of replicating certain key features of the U.S. economy. The autocorrelations of output stand out the most. The difficulty in replicating the lags of output may be a result of other driving forces in the real economy, such as animal spirits or price rigidities. Nuno Barrau (2008) also provides a variance decomposition of the observed variables resulting from the shocks. This decomposition illustrates that the driving force of business cycles in the simulated economy is the capital utilization shocks, not the productivity shocks.

In the view of Schumpeter, business cycles were caused by irregular arrival of innovations followed by monopolistic production. But, the variance decomposition does not support this conclusion. This indicates that business cycles may be a result of other economic processes. Another school of macroeconomic research suggests that neo-Keynesian processes drive business cycles. These processes are results of nominal rigidities in the economy. Creating a capital utilization framework from these principles may make the model more realistic.

4.3.3 Variations of the Model

The simulations undertaken so far are dependent upon the ability to determine the value of the parameters correctly. An economic model must be robust to changes in the model parameters because the values of these parameters are subject to considerable uncertainty. In this section, two parameter changes will be tested. First, a change in the risk-aversion coefficient, γ, will be examined. Then, a change in the output elasticity of the intermediate goods, α, will be analyzed. By changing these parameters we can see whether the reasonable predictions of the model are sensitive to parameter choice.

Table 4.5: Simulations with Changes in Gamma

	$\gamma = 2.479$	$\gamma = 1.5$	$\gamma = 3.5$
Std. Deviations			
dy_t	0.0201	0.0202	0.0200
dc_t	0.0161	0.0158	0.0164
di_t	0.0779	0.0735	0.0801
dL_t	0.0027	0.0027	0.0027
Correlations			
(dy_t, dc_t)	0.9198	0.9141	0.9246
(dy_t, di_t)	0.8388	0.8530	0.8198
(dy_t, dL_t)	-0.0290	-0.0272	-0.0198
(di_t, dc_t)	0.6438	0.6542	0.6313
(dy_t, dy_{t-1})	0.1325	0.1351	0.1100
(dy_t, dy_{t-1})	0.0898	0.1134	0.0946

The risk-aversion coefficient was initially determined through the estimation procedure. Thus, this value was based on the real economic data. A change in the risk-aversion coefficient would change the intertemporal elasticity of substitution. This would change the sensitivity of the consumption-saving decision to changes in interest rates. The initial value of the risk-aversion coefficient was 2.479. The values of 1.5 and 3.5 were tested to see if the simulations varied. The simulation results, reported in Table 4.5, did not vary dramatically with the changed values. This indicates the model's robustness with respect to a change in consumption preferences.

A change in the output elasticity of the intermediate good, α, was also tested. This parameter was originally calibrated from microeconomic evidence. Thus, the value may not be exact. The original value was 0.35; values of 0.2 and 0.5 were used to evaluate the robustness with respect to alpha. An increase in the output elasticity of the intermediate good increases the productivity of the intermediate good in the production of final goods. It also will decrease the share of labor and technology that goes into the final output. A higher alpha should result in a more volatile output because it will depend more on the intermediate good, which is subject to shocks. These new values had a dramatic effect on the simulations, unlike a change in the risk-aversion coefficient.

Table 4.6: Simulations with Changes in Alpha

	$\alpha = 0.35$	$\alpha = 0.2$	$\alpha = 0.5$
Std. Deviations			
dy_t	0.0201	0.0134	0.0276
dc_t	0.0161	0.0128	0.0190
di_t	0.0779	0.0999	0.0709
dL_t	0.0027	0.0027	0.0026
Correlations			
(dy_t, dc_t)	0.9198	0.9118	0.9206
(dy_t, di_t)	0.8388	0.5902	0.9305
(dy_t, dL_t)	-0.0290	-0.0475	-0.0330
(di_t, dc_t)	0.6438	0.3973	0.7693
(dy_t, dy_{t-1})	0.1325	0.3565	0.0361
(dy_t, dy_{t-1})	0.0898	0.3251	0.0215

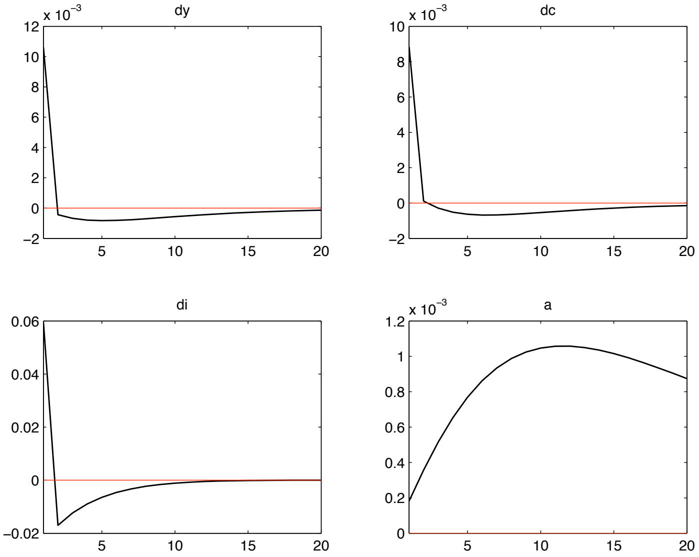

Figure 4.5: IRF for a Capital Utilization Shock, u, with $\alpha = 0.2$

Changing alpha dramatically changed the simulation results. Examining the standard deviations in Table 4.6 several things stick out. First, output volatility dramatically increases with an increase in alpha. As discussed, this is a result of being more dependent upon the intermediate good for production. There was also a large change in the standard deviation of investment. This result is a little more peculiar.

A decrease in alpha increases the standard deviation much more than the increase in alpha decreases it. A lower alpha requires a higher level of capital and utilization of that capital. Because capital utilization is stochastic, the economy will respond to the shocks by changing investment. Examining Figures 4.5 and 4.6, we can see that a positive capital utilization shock affects investment more with a lower alpha. The higher investment also corresponds to a greater deviation from zero for output and consumption, not seen with the higher alpha. The volatility of investment with respect to output and consumption can also be seen in the correlations of those variables. At a lower alpha, these relationships are much weaker.

The innovation shocks reveal more about a change in alpha. With a higher alpha, the shocks are smoother and less drastic. In Figure 4.8 we can see that the shock

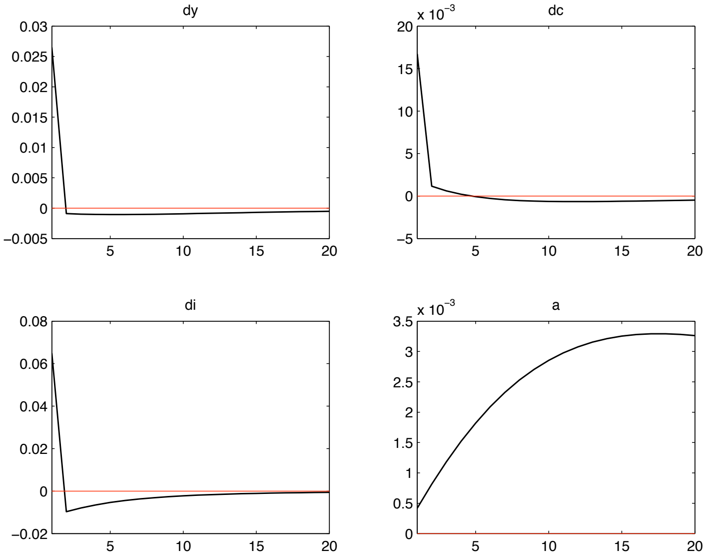

Figure 4.6: IRF for a Capital Utilization Shock, u, with $\alpha = 0.5$

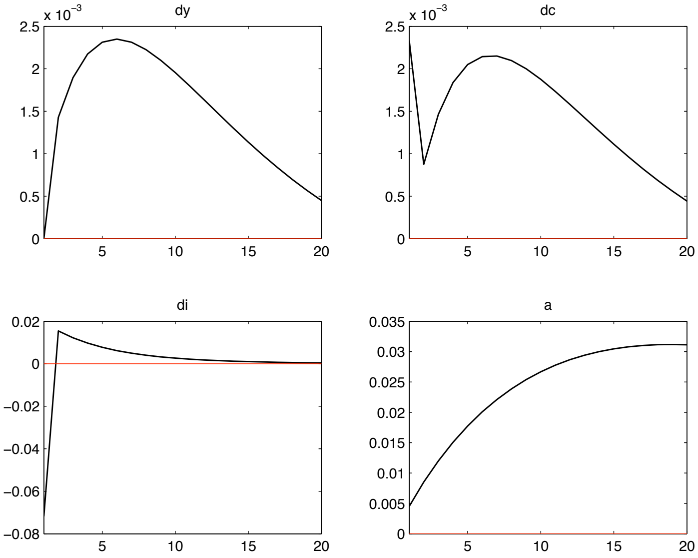

Figure 4.7: IRF for an Innovation Shock, λ, with $\alpha = 0.2$

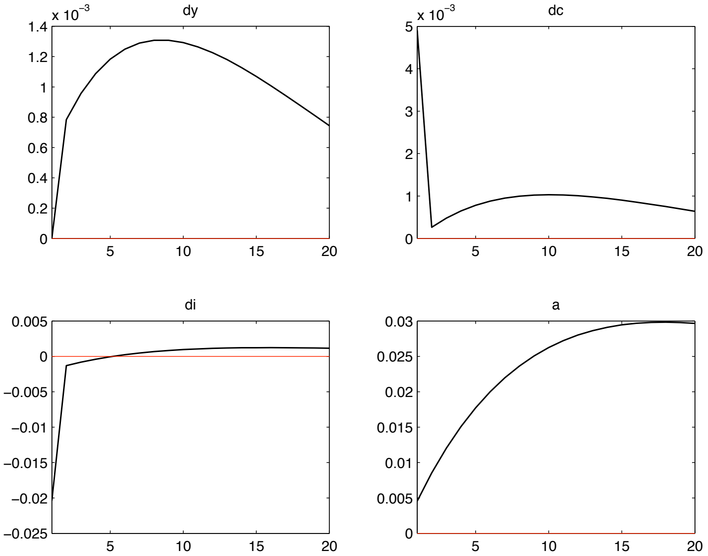

Figure 4.8: IRF for an Innovation Shock, λ, with $\alpha = 0.5$

takes awhile to affect the distance to the technological frontier, a, causing a smoother adjustment in the other observed variables. The adjustment of the technological frontier takes longer with a higher alpha because of less investment in research following the shock. In a sense, with the lower alpha, investors hope to take immediate advantage of the increases in the productivity of research tools. This advantage will pay a greater return if an innovation is found, but it is more difficult.

Finally, it is interesting to note the autocorrelations of output. A lower alpha causes these values to jump dramatically. Production of final goods depends more on the non-stochastic input, labor. This results in a more consistent output as it takes longer for shocks to affect output. These changes illustrate the effects of different production processes affecting the output elasticity of certain inputs. This certain model is not particularly robust with respect to these changes.

Conclusion

Schumpeterian creative destruction models like the one presented here attempt to improve upon real business cycle models. In RBC models the driving force of the economy is often an exogenous autoregressive stochastic process creating technological shocks. Creative destruction models build an economy based on the same theory. But, rather than an exogenous process, it has endogenous productivity increases. These are often driven in part by some kind of exogenous innovation shock. With the goal of mirroring the real economy, a richer model was thought to be taking a step towards that goal. Unfortunately, the more complex model did not make an improvement over the standard RBC model. This setback could also be viewed with optimism.

The model created has more realistic assumptions with more concrete mirco-foundations. Thus, the model was made stonger while not losing any prediction power. This is a step in the right direction. The model may be further improved by adding neo-Keynesian features. One neo-Keynesian feature of particular interest is labor market inefficiencies. Both RBC and creative destruction models like this one have difficulty replicating the relationship between output and employment. In Section 1.1.3 some variations of the basic RBC model were proposed. Similar components could be added to the creative destruction model in order to properly model employment dynamics. The most appealing to add would be indivisible labor, which is a better representation of the real economy. Many other neo-Keynesian features have been added to RBC models. These could also be developed in a creative destruction framework. Complex models of these sorts have been limited by the ability to evaluate them in the past.

As more advanced econometric methods and computational processes are developed, more advanced models can be developed and tested. The popularity of DSGE models resulted from the advancement of computers and this process will most likely continue. Because DSGE models are often constructed using microeconomic foundations, they will continue to be developed further.

A large part of this thesis was learning the process of developing, solving, and simulating a DSGE model. The thesis required becoming proficient in using several mathematical techniques common in macroeconomics, such as solving a model and linearizing it. In simulating the model, the knowledge of other useful tools was required. The process of determining parameters values either through calibration or estimation was used. And finally, the appropriate computer programs used to simulate the model were acquired and utilized. After learning all of these skills, the model and simulation of that model was able to be successfully replicated. Much can still be learned about the building and simulation of DSGE models in general. The skills acquired will be invaluable for future macroeconomic research.

Appendix A

Log-Linearized Model

In order to approximate the solution to a non-linear dynamic model, the model is first linearized. The process most often used is log-linearization. The model is linearized around the steady state. This allows the deviations from the steady state to be measured and analyzed. There are several methods of log-linearization, the calculations here use the method found in Uhlig (1995). This method takes advantage of the relationship between the log-linearized variables. The other methods usually involve taking first-order Taylor approximations. Both of these results produce the same solution, and each have their advantage depending on the equation.

Define a variable $\tilde{x}_t = \log(\frac{x_t}{x})$ where x is the value of x_t in the steady state. Thus, $\tilde{x}_t$ is the log deviations from the steady state. In order to linearize the model, first replace each variable $x_t = x\mathrm{e}^{\tilde{x}_t}$. If the equation only involves products, then take the logarithm of both sides, or use the following substitutions:

$$
\begin{aligned}
\mathrm{e}^{\tilde{x}_t + a\tilde{y}_t} &\approx 1 + \tilde{x}_t + a\tilde{y}_t \\
\tilde{x}_t\tilde{y}_t &\approx 0 \\
\mathbb{E}_t[a\mathrm{e}^{\tilde{x}_{t+1}}] &\approx \mathbb{E}_t[a\tilde{x}_{t+1}]
\end{aligned}
$$

The equations are then simplified by substituting in the steady state values and canceling. After substituting and simplifying, the linearization process is complete.

The twelve variables in the system describe the model. There are eight endogenous variables $(\tilde{r}_t, \tilde{a}_t, \tilde{n}_t, \tilde{q}_t, \tilde{y}_t, \tilde{i}_t, \tilde{c}_t, \tilde{k}_t)$ and four exogenous variable $(\tilde{u}_t, \tilde{\lambda}_t, \tilde{\xi}_t, \tilde{G}_t^L)$. Each of the twelve equations defining the solution to the system is linearized. Equations 3.3.4 and 3.6.3 will be used to illustrate the log-linearization process.

$$\hat{y}_t = (u_t \hat{k}_t)^\alpha$$

First the substitution for the log-linearized variable is made, then the equation is simplified:

$$\begin{aligned} y\mathrm{e}^{\tilde{y}_t} &= (u\mathrm{e}^{\tilde{u}_t}\hat{k}\mathrm{e}^{\tilde{k}_t})^\alpha \\ y\mathrm{e}^{\tilde{y}_t} &= (u\hat{k})^\alpha(\mathrm{e}^{\tilde{u}_t+\tilde{k}_t})^\alpha \end{aligned}$$

Since in the steady state $y = (u\hat{k})^\alpha$ that can be cancelled out of the equation. Then taking logs of both sides yields:

$$\tilde{y}_t = \alpha(\tilde{u}_t + \tilde{k}_t) \tag{A.0.1}$$

If the equation contains addition or subtraction, the process becomes more complex. Equation 3.6.3 will be used as an example. The process is similar, but the quick substitutions are used rather than taking the log of both sides.

$$\begin{aligned} a_{t+1} &= \frac{(1-a_t)n_t + a_t}{1+g} \\ a\mathrm{e}^{\tilde{a}_{t+1}} &= \frac{n\mathrm{e}^{\tilde{n}_t} - a\mathrm{e}^{\tilde{a}_t}n\mathrm{e}^{\tilde{n}_t} + a\mathrm{e}^{\tilde{a}_t}}{1+g} \end{aligned}$$

Using the building block suggested in Uhlig, the equation becomes

$$a(1+\tilde{a}_{t+1})(1+g) = n(1+\tilde{n}_t) - an(1+\tilde{a}_t+\tilde{n}_t) + a(1+\tilde{a}_t)$$

The steady state relationship $n = a(n+g)$ is substituted in for n giving:

$$\tilde{a}_{t+1}(1+g) = ag + an\tilde{n}_t + ag\tilde{n}_t + a\tilde{a}_t - an\tilde{a}_t$$

And simplifying yields:

$$\tilde{a}_{t+1} = \frac{g}{1+g}\tilde{n}_t + \frac{(1-n)}{1+g}\tilde{a}_t \tag{A.0.2}$$

The dynamics of the model is defined by 12 equations with eight endogenous variables and the four stochastic exogenous shocks. These equations are linearized through the same procedure discussed:

$$\tilde{r}_t = -\frac{(1-\alpha)(r+\delta)}{r}\tilde{k}_t + \frac{\alpha(r+\delta)}{r}\tilde{u}_t \tag{A.0.3}$$

$$\tilde{n}_t = \alpha\rho_u\tilde{u}_t + \alpha\tilde{k}_{t+1} - \frac{a}{1-a}\tilde{a}_t - \frac{r}{1+r}\mathbb{E}_t[\tilde{r}_{t+1}] + \tilde{G}^L_t - \tilde{\lambda}_t \tag{A.0.4}$$

$$\tilde{q}_t = -\tilde{a}_t + 2\tilde{n}_t + \tilde{\lambda}_t \tag{A.0.5}$$

$$\tilde{i}_t = \frac{\hat{y}}{\hat{i}}\tilde{y}_t - \frac{\hat{c}}{\hat{i}}\tilde{c}_t - \frac{\hat{q}}{\hat{i}}\tilde{q}_t - \frac{\xi}{\hat{i}}\tilde{\xi}_t \tag{A.0.6}$$

$$\tilde{c}_t = \tilde{a}_{t+1} - \tilde{a}_t + \mathbb{E}_t[\tilde{c}_{t+1}] - \frac{r}{\gamma(1+r)}\mathbb{E}_t[\tilde{r}_{t+1}] - \frac{1}{\gamma}\tilde{G}^L_t \tag{A.0.7}$$

$$\tilde{k}_{t+1} = -\tilde{a}_{t+1} + \tilde{a}_t - \tilde{G}^L_t + \frac{\hat{i}}{\hat{k}(1+g)G^L}\tilde{i}_t + \frac{(1-\delta)}{(1+g)G^L}\tilde{k}_t \tag{A.0.8}$$

The four exogenous shocks follow the same stochastic structure:

$$\tilde{x}_{t+1} = \rho_x\tilde{x}_t + \sigma_x\epsilon_{x,t+1} \tag{A.0.9}$$

for an exogenous variable $\tilde{x}_t$.

Appendix B

Sample Simulation File

Below is the mod-file used to simulate the model. A mod-file is the input into Dynare which it uses to simulate the economy.

```
//Endogenous variables
var r, a, n, q, y, i, c, k, u, lambda, xi, GL, dy, dL, dc, di;

//Exogenous stochastic vars(vars that will be shocked)
varexo epsilonu, epsilonL, epsilonlambda, epsilonxi;

//Parameters
parameters alpha, beta, delta, g, GLbar, gamma, ubar, nbar,
lambdabar,
    rholambda, rhou, rhoxi, rhoL, sigmalambda, sigmau, sigmaxi,
sigmaL;

//Calibrated Parameters
alpha=0.85;
beta=0.99;
g=0.0191;
GLbar=1.0135;

//Estimated Parameters(Taken from Paper)
nbar = 0.017;
gamma=2.479;
delta=0.115;
rholambda=0.946;
rhou=0.783;
rhoxi=0.949;
rhoL=0.928;
sigmalambda=0.247;
sigmau=0.053;
sigmaxi=0.289;
sigmaL=0.001;

//Steady State Values
lambdabar = 1.03897;
ubar = .8;
```

```
model (linear);
//Calculation of Steady State Values
 # abar = nbar/(nbar+g);
 # qhat = (lambdabar*(nbar^2))/(2*abar);
 # rbar = ((GLbar*((1+g)^gamma))/beta)-1;
 # khat = (((alpha^2)*(ubar^alpha))/(rbar+delta))^(1/(1-alpha));
 # ihat = ((1+g)*GLbar-(1-delta))*khat;
 # yhat = (ubar*khat)^alpha;
 # xibar = -0.016*yhat;
 # chat = yhat-ihat-qhat-xibar;

//Linearized Model
    r = -(((1-alpha)*(rbar+delta)*k(-1))/rbar)+
((alpha*(rbar+delta)*u)/rbar);
    a = (((1-nbar)*a(-1))/(1+g))+((g*n)/(1+g));
    n = alpha*rhou*u+alpha*k-((abar*a(-1))/(1-abar))-
((rbar*r(+1))/(1+rbar))+GL+lambda;
    q = -a(-1)+2*n-lambda;
    y = alpha*k(-1)+alpha*u;
    i= (yhat*y)/ihat-(chat*c)/ihat-(qhat*q)/ihat-(xibar*xi)/ihat;
    c = a-a(-1)+c(+1)-((rbar*r(+1))/(gamma*(1+rbar)))+(GL/gamma);
k = -a+a(-1)-GL+((ihat*i)/(khat*(1+g)*GLbar))+(((1-delta)*k(-
1))/((1+g)*GLbar));
    u = rhou*u(-1)+sigmau*epsilonu;
    lambda = rholambda*(lambda(-1))+ sigmalambda*epsilonlambda;
    GL = rhoL*(GL(-1))+sigmaL*epsilonL;
    xi = rhoxi*(xi(-1))+sigmaxi*epsilonxi;
    dy = log(1+g)+(a(-1)-a(-2))+(y-y(-1));
    dL = log(GLbar)+GL(-1);
    dc = log(1+g)+(a(-1)-a(-2))+(c-c(-1));
    di = log(1+g)+(qhat/(qhat+ihat))*(q-q(-
1))+(ihat/(qhat+ihat))*(i-i(-1));
end;

//Steady State Values
initval;
    r = 0;
    a = 0;
    n = 0;
    q = 0;
    y = 0;
    i = 0;
    c = 0;
    k = 0;
    u = 0;
    lambda = 0;
    xi = 0;
    GL = 0;
    dy = 0.00821680159;
    dL = 1.01932375;
    dc = 0.00821680159;
    di = 0.00821680159;
end;
steady;
check;
shocks;
    var epsilonu = 1;
    var epsilonxi = 1;
    var epsilonL = 1;
    var epsilonlambda = 1;
end;

stoch_simul(periods=10000, irf=20) dy dc di dL a;
```

Appendix C

Dynare Simulation Process

This appendix comes verbatim from Griffoli (2007), the Dynare user manual, which describes the mathematics behind the simulation process:

In this section, we shall briefly overview the perturbation methods employed by Dynare to solve DSGE models to a first order approximation. The second order follows very much the same approach, although at a higher level of complexity. The summary below is taken mainly from Michel Juillard's presentation "Computing first order approximations of DSGE models with Dynare," which you should read if interested in particular details, especially regarding second order approximations (available on Michel Juillard's website).

To summarize, a DSGE model is a collection of first order and equilibrium conditions that take the general form:

$$\mathbb{E}_t\{f(y_{t+1}, y_t, y_{t-1}, u_t)\} = 0$$
$$\mathbb{E}_t(u_t) = 0$$
$$\mathbb{E}_t(u_t u_t') = \sum u$$

and where:

y : vector of endogenous variables of any dimension

u : vector of exogenous stochastic shocks of any dimension

The solution to this system is a set of equations relating variables in the current period to the past state of the system and current shocks, that satisfy the original system. This is what we call the policy function. Sticking to the above notation, we can write this function as:

$$y_t = g(y_{t-1}, u_t)$$

Then, it is straightforward to re-write y_{t+1} as

$$\begin{aligned} y_{t+1} &= g(y_t, u_{t+1}) \\ &= g(g(y_{t-1}, u_t), u_{t+1}) \end{aligned}$$

We can then define a new function F, such that:

$$F(y_{t-1}, u_t, u_{t+1}) = f(g(g(y_{t-1}, u_t), u_{t+1}), g(y_{t-1}, u_t), y_{t-1}, u_t)$$

which enables us to rewrite our system in terms of past variables, and current and future shocks:

$$\mathbb{E}_t \left[F(y_{t-1}, u_t, u_{t+1})\right] = 0$$

We then venture to linearize this model around a steady state defined as:

$$f(\bar{y}, \bar{y}, \bar{y}, 0) = 0$$

having the property that:

$$\bar{y} = g(\bar{y}, 0)$$

The first order Taylor expansion around $\bar{y}$ yields:

$$\begin{aligned} &\mathbb{E}_t \left\{F^{(1)}(y_{t-1}, u_t, u_{t+1})\right\} = \\ &\quad \mathbb{E}_t \Big[f(\bar{y}, \bar{y}, \bar{y}) + f_{y_+} \left(g_y \left(g_y \hat{y} + g_u u\right) + g_u u'\right) \\ &\quad + f_{y_0} \left(g_y \hat{y} + g_u u\right) + f_{y_-} \hat{y} + f_u u \Big] \\ &\quad = 0 \end{aligned}$$

with $\hat{y} = y_{t-1} - \bar{y}$, $u = u_t$, $u' = u_{t+1}$, $f_{y_+} = \frac{\partial f}{\partial y_{t+1}}$, $f_{y_0} = \frac{\partial f}{\partial y_t}$, $f_{y_-} = \frac{\partial f}{\partial y_{t-1}}$, $f_u = \frac{\partial f}{\partial u_t}$, $g_y = \frac{\partial g}{\partial y_{t-1}}$, $g_u = \frac{\partial g}{\partial u_t}$.

Taking expectations (we're almost there!):

$$\begin{aligned} &\mathbb{E}_t \left\{F^{(1)}(y_{t-1}, u_t, u_{t+1})\right\} = \\ &\quad f(\bar{y}, \bar{y}, \bar{y}) + f_{y_+} \left(g_y \left(g_y \hat{y} + g_u u\right)\right) \\ &\quad + f_{y_0} \left(g_y \hat{y} + g_u u\right) + f_{y_-} \hat{y} + f_u u \Big\} \\ &= \left(f_{y_+} g_y g_y + f_{y_0} g_y + f_{y_-}\right) \hat{y} + \left(f_{y_+} g_y g_u + f_{y_0} g_u + f_u\right) u \\ &= 0 \end{aligned}$$

As you can see, since future shocks only enter with their first moments (which are zero in expectations), they drop out when taking expectations of the linearized equations. This is technically why certainty equivalence holds in a system linearized to its first order. The second thing to note is that we have two unknown variables in the above equation: g_y and g_u each of which will help us recover the policy function g.

Since the above equation holds for any $\hat{y}$ and any u, each parenthesis must be null and we can solve each at a time. The first, yields a quadratic equation in g_y, which we can solve with a series of algebraic trics that are not all immediately apparent (but detailed in Michel Juillards presentation). Incidentally, one of the conditions that comes out of the solution of this equation is the Blanchard-Kahn condition: there must be as many roots larger than one in modulus as there are forward-looking variables in the model. Having recovered g_y, recovering g_u is then straightforward from the second parenthesis.

Finally, notice that a first order linearization of the function g yields:

$$y_t = \bar{y} + g_y \hat{y} + g_u u$$

And now that we have g_y and g_u, we have solved for the (approximate) policy (or decision) function and have succeeded in solving our DSGE model. If we were interested in impulse response functions, for instance, we would simply iterate the policy function starting from an initial value given by the steady state.

The second order solution uses the same perturbation methods as above (the notion of starting from a function you can solve - like a steady state - and iterating forward), yet applies more complex algebraic techniques to recover the various partial derivatives of the policy function. But the general approach is perfectly isomorphic. Note that in the case of a second order approximation of a DSGE model, the variance of future shocks remains after taking expectations of the linearized equations and therefore affects the level of the resulting policy function.

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